Gravity and Quantum Fields in Discrete Space-Times

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Based on work in collaboration with

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Overview

- 6D space-times with 2 extra dimensions.
- Extra dimensions form a 2D curved disk, e.g., a part of a sphere.
- Discrete disk, 4D space continuous.
- Formal basis for effective theories of massive 4D gravitons.
  - Deconstruction, transverse lattice. Arkani-Hamed, Cohen, Georgi; Hill, Pokorski, Wang
  - Discrete gravitational dimensions. Arkani-Hamed, Georgi, Schwartz
- Special discretisations.
  - 6D action ▶ Action for massive 4D gravitons.
  - Mass eigenstates, dependence of mass eigenvalues on the disk parameters.
- 6D Dirac fermion on discrete disk ▶ massive 4D fermions.
  - Mass eigenstates and eigenvalues.
  - Comparison with gravitons.
  - Generation of light SM fermion masses.
- Refined scenario with "warping", mass spectrum.

Extra-dimensional curved disk in the continuum

- Space-time: 4D $\times$ 2D curved disk, 4D Metric $g_{\mu\nu}$:

$$\mathrm{d}s^2 = g_{\mu\nu}(x^M)\mathrm{d}x^\mu \mathrm{d}x^\nu - \frac{1}{1-er^2}\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2,$$

- Coordinates on the disk:
  - radial $r = 0...L$.
  - angular $\varphi = 0...2\pi$.
  - disk boundary: $r = L$.
- Curvature:
  - flat: $e = 0$.
  - spherically curved: $e > 0$.
  - hyperbolically curved: $e < 0$.
  - curvature radius: $1/\sqrt{e}$.

Einstein's equations

- Flat 4D space: $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.
- Einstein tensor:

$$G_{MN} = -e \cdot \eta_{\mu\nu}.$$

- Include modified cosmological term that yields $e \cdot \eta_{\mu\nu}$:
- 6D Action:

$$S = M_6^4 \int \text{d}^6x \, \sqrt{|g|} \, [R - 2\lambda \cdot (g^{AB} n_{AB})]$$

$$\text{with} \quad n_{AB} := \text{diag}(0, 0, 0, 0, \frac{1}{2}g_{55}, \frac{1}{2}g_{66}).$$

- Variation $\delta S / \delta g^{AB} = 0$:

$$G_{MN} + g_{\mu\nu}\lambda = 0.$$

- Solve Einstein's equations for the flat 4D case with $\lambda = e$.

6D action

- Split action:

$$S = M_6^4 \int \mathrm{d}^6x \, \sqrt{|g|} R = S_{\text{kin}} + S_{\text{mass}} + \cdots \, .$$

$$\begin{aligned} S_{\text{kin}} &= M_6^4 \int \mathrm{d}^6x \sqrt{|g|}\, R_{4\text{D}}, \\ S_{\text{mass}} &= M_6^4 \int \mathrm{d}^6x \sqrt{|g|} \Big[-\frac{1}{4} g^{55} g_{\mu\nu,5} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) g_{\alpha\beta,5} \Big] \\ &+ M_6^4 \int \mathrm{d}^6x \sqrt{|g|} \Big[-\frac{1}{4} g^{66} g_{\mu\nu,6} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) g_{\alpha\beta,6} \Big]. \end{aligned}$$

- Usual 4D Ricci scalar: $R_{4\text{D}}$.
  Determinant of g_{MN}: $g = g_4 \cdot g_{55} \cdot g_{66}$, $g_4 = \det g_{\mu\nu}$.

- $g_{55} = 1/(1 - er^2)$, $g_{66} = r^2$:

$$\begin{aligned} S_{\text{mass}} &= M_6^4 \int \mathrm{d}^4x \int \mathrm{d}\varphi \, \mathrm{d}r \, \sqrt{|g_4|} \left[+ \frac{1}{4} r \sqrt{1 - er^2}\, \partial_r g_{\mu\nu} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) \partial_r g_{\alpha\beta} \right] \\ &+ M_6^4 \int \mathrm{d}^4x \int \mathrm{d}\varphi \, \mathrm{d}r \, \sqrt{|g_4|} \left[+ \frac{1}{4} \frac{1}{r\sqrt{1 - er^2}}\, \partial_\varphi g_{\mu\nu} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) \partial_\varphi g_{\alpha\beta} \right]. \end{aligned}$$

Gravitons

- Expand metric $g_{\mu\nu}$ around flat 4D space $\eta_{\mu\nu}$ ▶ graviton fields $h_{\mu\nu}$:

$$g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}.$$

- S_{mass} in second order in $h_{\mu\nu}$:

$$\begin{aligned} S_{\text{mass}} &\to M_6^4 \int \mathrm{d}^4x \int \mathrm{d}\varphi\, \mathrm{d}r \left[+\frac{1}{4} r\sqrt{1-er^2}\, \partial_r h_{\mu\nu} (\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}) \partial_r h_{\alpha\beta} \right] \\ &+ M_6^4 \int \mathrm{d}^4x \int \mathrm{d}\varphi\, \mathrm{d}r \left[+\frac{1}{4} \frac{1}{r\sqrt{1-er^2}}\, \partial_\varphi h_{\mu\nu} (\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}) \partial_\varphi h_{\alpha\beta} \right]. \end{aligned}$$

- We do not consider h_{5M}, h_{6M}.
- Graviton kinetic terms in second order:

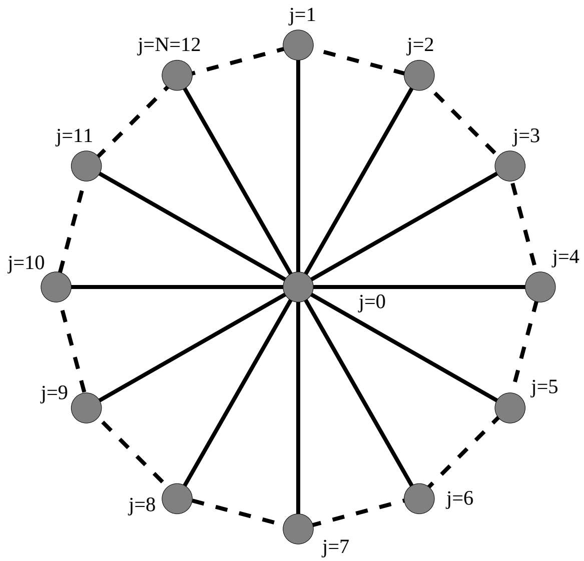
$$\sqrt{|g|}\, R_{4D} \to \sqrt{|g_{55}\, g_{66}|} \frac{1}{4} (\partial^\mu h^{\nu\kappa} \partial_\mu h_{\nu\kappa} - \partial^\mu h \partial_\mu h - 2h^\mu h_\mu + 2h^\mu \partial_\mu h)$$

$$\text{with} \quad h := h^\mu{}_\mu, \quad h_\nu := \partial^\mu h_{\mu\nu}.$$

Discretisation

- One lattice site in the centre, N sites on boundary at equal distance. Only two points in radial direction.

Arkani-Hamed, Cohen, Georgi; Witten; ...



- Radial lattice spacing:

$$\Delta r = L.$$

- Angular lattice spacing:

$$\Delta\varphi = \frac{2\pi}{N}.$$

- Graviton field in the centre: $h^0_{\mu\nu}$, on the boundary: $h^j_{\mu\nu}$ with $j = 1 \ldots N$.

- Discretisation prescription:

$$\begin{aligned}
\partial_r h(\varphi^i) &\rightarrow \frac{(h^i - h^0)}{\Delta r},\\
\partial_\varphi h(\varphi^i) &\rightarrow \frac{(h^{i+1} - h^i)}{\Delta\varphi},\\
\int \mathrm{d}r\, f(r) &\rightarrow \sum_{r=L} \Delta r \cdot f(L) = \Delta r \cdot f(L),\\
\int \mathrm{d}\varphi\, f(\varphi) &\rightarrow \sum_{i=1}^{N} \Delta\varphi \cdot f(\varphi^i).
\end{aligned}$$

  Evaluate $f(L)$ just on the boundary.

- Apply this to S_{mass}:

$$\begin{aligned}
S_{\text{mass}} &= M_6^4 \int \mathrm{d}^4x \int \mathrm{d}\varphi\, \mathrm{d}r \left[+ \frac{1}{4} r\sqrt{1-er^2}\, \partial_r h_{\mu\nu}(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})\partial_r h_{\alpha\beta} \right]\\
&+ M_6^4 \int \mathrm{d}^4x \int \mathrm{d}\varphi\, \mathrm{d}r \left[+ \frac{1}{4} \frac{1}{r\sqrt{1-er^2}}\, \partial_\varphi h_{\mu\nu}(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})\partial_\varphi h_{\alpha\beta} \right].
\end{aligned}$$

- Result:

$$S_{\text{mass}} \rightarrow M_6^4 \int \mathrm{d}^4x \sum_{i=1}^{N} \Delta\varphi\Delta r$$
$$\times \left[+\frac{1}{4} L\sqrt{1-eL^2} \cdot \frac{h^i_{\mu\nu} - h^0_{\mu\nu}}{\Delta r} (\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}) \frac{h^i_{\alpha\beta} - h^0_{\alpha\beta}}{\Delta r} \right.$$
$$\left. + \frac{1}{4} \frac{1}{L\sqrt{1-eL^2}} \cdot \frac{h^{i+1}_{\mu\nu} - h^i_{\mu\nu}}{\Delta\varphi} (\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}) \frac{h^{i+1}_{\alpha\beta} - h^i_{\alpha\beta}}{\Delta\varphi} \right].$$

- Fierz-Pauli graviton mass term:

$$M_4^2 \int \mathrm{d}^4x \, m^2 \cdot h_{\mu\nu}(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}) h_{\alpha\beta}.$$

- S_{mass} in Fierz-Pauli form (non-diagonal):

$$S_{\text{mass}} \rightarrow M_4^2 \int \mathrm{d}^4x \sum_{i=1}^{N}$$
$$\times \left[+m_\star^2 \cdot (h^i_{\mu\nu} - h^0_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(h^i_{\alpha\beta} - h^0_{\alpha\beta}) \right.$$
$$\left. + m^2 \cdot (h^{i+1}_{\mu\nu} - h^i_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(h^{i+1}_{\alpha\beta} - h^i_{\alpha\beta}) \right].$$

- S_{mass} in Fierz-Pauli form (non-diagonal):

$$\begin{aligned} S_{\mathsf{mass}} \;\rightarrow\; & M_4^2 \int \mathrm{d}^4x \sum_{i=1}^{N} \\ \times \; & \Big[+m_\star^2 \cdot (h^i_{\mu\nu} - h^0_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(h^i_{\alpha\beta} - h^0_{\alpha\beta}) \\ & +m^2 \cdot (h^{i+1}_{\mu\nu} - h^i_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(h^{i+1}_{\alpha\beta} - h^i_{\alpha\beta})\Big] . \end{aligned}$$

- Mass scales depend on M_4:

$$m_\star^2 := \frac{M_6^4}{M_4^2} \cdot \frac{1}{4} \cdot \frac{2\pi}{N} \cdot \sqrt{1 - eL^2},$$

$$m^2 := \frac{M_6^4}{M_4^2} \cdot \frac{1}{4} \cdot \frac{N}{2\pi} \cdot \frac{1}{\sqrt{1 - eL^2}}.$$

- Ratio independent of M_4:

$$\frac{m_\star^2}{m^2} = \frac{(2\pi)^2}{N^2}(1 - eL^2).$$

- Can be tuned by changing the disk parameters: e, L, N.

4D Planck scale M_4

- Discretise kinetic terms:

$$S_{4\mathrm{D}} = M_6^4 \int \mathrm{d}^6x \sqrt{|g|} R_{4\mathrm{D}} = M_6^4 \underbrace{\int \mathrm{d}\varphi\, \mathrm{d}r\, \sqrt{|g_{55} g_{66}|}}_{\text{proper area } A} \int \mathrm{d}^4x\, \sqrt{|g_4|}\, R_{4\mathrm{D}}.$$

- Naive discretisation of $\int \mathrm{d}r$ yields no kinetic term for the centre site $r = 0$ since

$$\sqrt{|g|} \propto \sqrt{|g_{66}|} = r = 0.$$

- Disk has constant curvature ▶ M_4 constant on all sites.
  $R_{4\mathrm{D}}$ terms of all $N + 1$ sites have the form

$$M_4^2 \sum_{i=0}^{N} \int \mathrm{d}^4x\ \sqrt{|g_4|} R_{4\mathrm{D}}\Big|_{\text{site } i} = M_4^2 \sum_{i=0}^{N} \int \mathrm{d}^4x\, \sqrt{|g_4|} R_{4\mathrm{D}}.$$

- We find

$$M_6^4 \underbrace{\int \mathrm{d}\varphi\, \mathrm{d}r\, \sqrt{|g_{55}g_{66}|}}_{\text{proper area } A} \int \mathrm{d}^4x\, \sqrt{|g_4|} R_{4\mathrm{D}} \;\to\; M_4^2 \sum_{i=0}^{N} \int \mathrm{d}^4x\; \sqrt{|g_4|} R_{4\mathrm{D}}\Big|_{\text{site } i}$$

$$= M_4^2 \sum_{i=0}^{N} \int \mathrm{d}^4x\, \sqrt{|g_4|} R_{4\mathrm{D}}$$

$$= M_4^2 (N+1) \int \mathrm{d}^4x\, \sqrt{|g_4|} R_{4\mathrm{D}}.$$

- 4D Planck scale on sites:

$$M_4^2 = \frac{M_6^4 A}{N+1}.$$

- 4D Planck scale in the continuum by integrating out the extra dimensions:

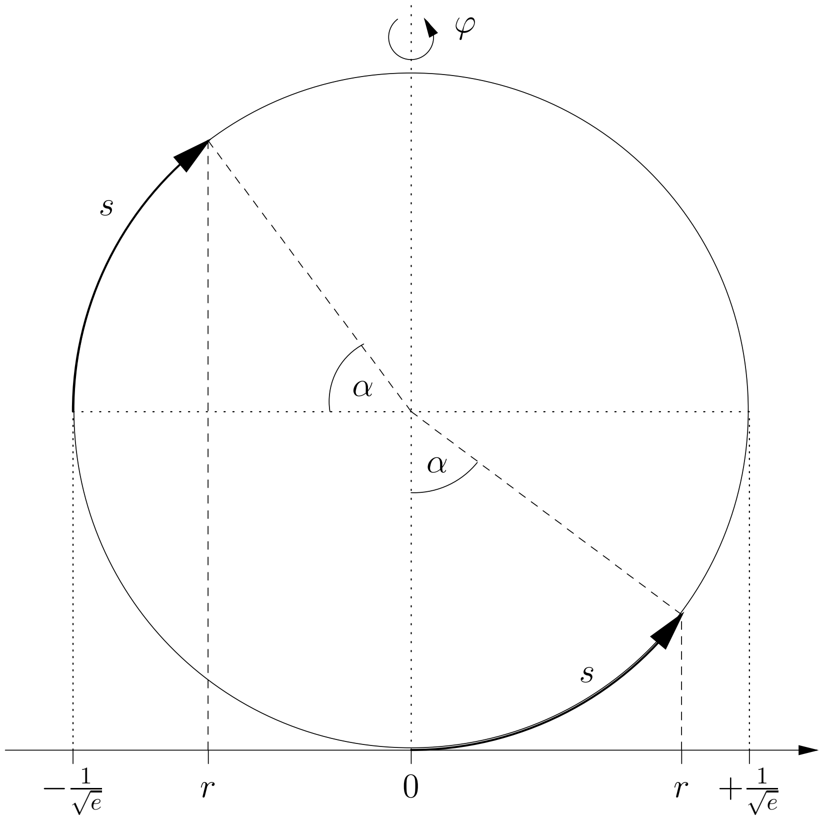
$$M_{\mathrm{Pl}}^2 = M_6^4 A = (N+1) M_4^2.$$

- Proper area A:

$$A := \int_0^{2\pi} \mathrm{d}\varphi \int_0^L \mathrm{d}r \sqrt{|g_{55} g_{66}|} = 2\pi \int_0^L \mathrm{d}r \frac{r}{\sqrt{1-er^2}} = \begin{cases} \frac{2\pi}{e}(1-\sqrt{1-eL^2}), & e>0 \\ \pi L^2, & e=0 \\ \frac{2\pi}{|e|}(\sqrt{1+|e|L^2}-1), & e<0. \end{cases}$$

- For $|e|L^2 \ll 1$ the area scales like $A = \pi L^2 + \mathcal{O}(eL^4)$.
- Spherically curved, $e > 0$: 2-sphere with radius $1/\sqrt{e}$.
  - First half sphere: $r \in [0, 1/\sqrt{e}]$; equator at $r = 1/\sqrt{e}$.
  - Second half: $r \in [-1/\sqrt{e}, 0]$;
    $r \longrightarrow 0$ with $r < 0$ means approaching the antipode.
  - Area of the whole sphere: $4\pi/e$.
  - Proper length $s(r)$:

$$s = \int_0^r \mathrm{d}r' \sqrt{|g_{55}|} = \int_0^r \mathrm{d}r' \frac{1}{\sqrt{1-er^2}} = \frac{1}{\sqrt{e}} \operatorname{arcsin}(\sqrt{e}r).$$



- Hyperbolically curved, $e < 0$:
  - $|e|L^2 \gg 1$: $A \approx 2\pi L/\sqrt{|e|}$.
  - Proper length $s(r) = \operatorname{arsinh}(\sqrt{|e|}r)/\sqrt{|e|}$.
- Flat, $e = 0$: $s(r) = r$.

Massive Gravitons

- Full 4D graviton action

$$
\begin{aligned}
S_{\text{graviton}} &= M_4^2 \sum_{i=0}^{N} \int \mathrm{d}^4x \, \frac{1}{4}(\partial^\mu h^{i\nu\kappa}\partial_\mu h^i_{\nu\kappa} - \partial^\mu h^i \partial_\mu h^i - 2h^{i\mu}h^i_\mu + 2h^{i\mu}\partial_\mu h^i) \\
&+ M_4^2 \sum_{i=1}^{N} \int \mathrm{d}^4x \\
&\times \Big[+m_\star^2 \cdot (h^i_{\mu\nu} - h^0_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(h^i_{\alpha\beta} - h^0_{\alpha\beta}) \\
&\quad +m^2 \cdot (h^{i+1}_{\mu\nu} - h^i_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(h^{i+1}_{\alpha\beta} - h^i_{\alpha\beta})\Big] \,.
\end{aligned}
$$

- Mass matrix

$$
M_{\mathrm{g}}^2 = m_\star^2 \begin{pmatrix} N & -1 & -1 & \cdots & -1 \\ -1 & 1 & & & \\ -1 & & 1 & & \\ \vdots & & & \ddots & \\ -1 & & & & 1 \end{pmatrix} + m^2 \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & -1 & & -1 \\ 0 & -1 & 2 & \ddots & \\ \vdots & & \ddots & \ddots & -1 \\ 0 & -1 & & -1 & 2 \end{pmatrix} .
$$

- Diagonalisation ▶ Mass eigenstates $H^j_{\mu\nu}$.
- Flat zero mode, $M_0^2 = 0$:

$$H^0_{\mu\nu} = \frac{1}{\sqrt{N+1}} \sum_{i=0}^{N} h^i_{\mu\nu}.$$

  Equally located on all sites.
- $N-1$ boundary modes, $p = 1 \ldots N-1$:

$$M_p^2 = m_\star^2 + 4m^2 \sin^2 \frac{\pi p}{N},$$

$$H^p_{\mu\nu} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[\sin(2\pi i \frac{p}{N}) + \cos(2\pi i \frac{p}{N})\right] h^i_{\mu\nu}.$$

  Typical discrete Kaluza-Klein mass spectrum shifted by $m_\star^2$ ▶ mass gap.
  Limit $m \ll m_\star$: masses almost degenerate.
- Heavy mode, $M_N^2 = (N+1)m_\star^2$:

$$H^N_{\mu\nu} = \frac{1}{\sqrt{N(N+1)}} \left[-N\, h^0_{\mu\nu} + \sum_{i=1}^{N} h^i_{\mu\nu}\right],$$

  Peaked on the centre site with equal support on the boundary sites.
- Application: Large mass gap for $m_\star \gg m$ ▶ hide large extra dimensions. Kim

Fermions on the discretised disk

- 6D Dirac action in the continuum:

$$S = \int \mathrm{d}^6x \, \sqrt{|g|} \left[\frac{1}{2} i \left(\overline{\Psi} G^A V_A^M \nabla_M \Psi - \overline{\nabla_M \Psi} V_A^M G^A \Psi \right) \right].$$

- Dirac spinor Ψ has 8 components.
- Six 8×8-γ-matrices G^A:
  - 4D: (with Pauli matrices σ^k)

$$\gamma^0 = \begin{bmatrix} 0 & 1_2 \\ 1_2 & 0 \end{bmatrix}, \; \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \; \gamma^5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}.$$

  - 5D: $\Gamma^0 = \gamma^0$, $\Gamma^k = \gamma^k$, $\Gamma^5 = i\gamma^5 = -(\Gamma^5)^\dagger$.
  - 6D:

$$\begin{aligned} G^0 &= \begin{bmatrix} 0 & 1_4 \\ 1_4 & 0 \end{bmatrix} = (G^0)^\dagger, \\ G^n &= \begin{bmatrix} 0 & \Gamma^0\Gamma^n \\ -\Gamma^0\Gamma^n & 0 \end{bmatrix} = -(G^n)^\dagger, n = 1, 2, 3, 5, \\ G^6 &= \begin{bmatrix} 0 & \Gamma^0 \\ -\Gamma^0 & 0 \end{bmatrix} = -(G^6)^\dagger. \end{aligned}$$

- Decouple spinor components in the action by

$$\Psi := G^6\Phi, \quad \Phi = (\Phi_a, \Phi_b)^\mathsf{T}$$

Φ_a, Φ_b: 4 components each.

- Dirac action diagonal:

$$i\left[\overline{\Phi_a}, \overline{\Phi_b}\right] \times \left[\begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{pmatrix} \partial_0 + \begin{pmatrix} -\gamma^k & 0 \\ 0 & \gamma^k \end{pmatrix} \partial_k + \begin{pmatrix} -i\gamma^5 & 0 \\ 0 & +i\gamma^5 \end{pmatrix} V_5\partial_5 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V_6\partial_6\right] \times \begin{bmatrix} \Phi_a \\ \Phi_b \end{bmatrix}.$$

"Φ_a" = "Φ_b" with negative energy.

- 6D: Spinor Φ has mass dimension $\frac{5}{2}$.
  Kinetic terms: Integrate out the extra dimensions, discretise as in the graviton case:

$$\int \mathsf{d}^6x\sqrt{|g|}\frac{1}{2}i\overline{\Phi_b}\gamma^\mu\partial_\mu\Phi_b \to \sum_{j=0}^{N}\frac{A}{N+1}\int \mathsf{d}^4x\frac{1}{2}i\overline{\Phi_b^j}\gamma^\mu\partial_\mu\Phi_b^j.$$

- Rescale spinor:

$$\chi := \Phi_b\sqrt{A/(N+1)}.$$

  ▶ χ has the usual 4D mass dimension $\frac{3}{2}$.

- Discretisation: (similar to gravitons)

$$\begin{aligned}
\partial_5\chi &\to (\chi^j - \chi^0)/\Delta r, \\
\partial_6\chi &\to (\chi^{j+1} - \chi^j)/\Delta\varphi, \\
\int \mathrm{d}r \int \mathrm{d}\varphi \cdot f &\to \sum_{j=1}^{N} \Delta r \Delta\varphi \cdot f^j .
\end{aligned}$$

- Action for $N+1$ 4D fermions: $\chi_{\mathsf{L,R}} = \frac{1}{2}(1 \mp \gamma^5)$

$$\begin{aligned}
S &= \sum_{j=0}^{N} \int \mathrm{d}^4x \frac{1}{2} i \left(\overline{\chi^j}\gamma^\mu \partial_\mu \chi^j - \overline{\partial_\mu \chi^j}\gamma^\mu \chi^j \right) \\
&- \sum_{j=1}^{N} \int \mathrm{d}^4x \cdot m_\star \left(\overline{\chi^j_{\mathsf{L}}}(\chi^j_{\mathsf{R}} - \chi^0_{\mathsf{R}}) + (\overline{\chi^j_{\mathsf{R}}} - \overline{\chi^0_{\mathsf{R}}})\chi^j_{\mathsf{L}} \right) \\
&- \sum_{j=1}^{N} \int \mathrm{d}^4x \cdot i \cdot m \left(\overline{\chi^j_{\mathsf{L}}}(\chi^{j+1}_{\mathsf{R}} - \chi^j_{\mathsf{R}}) - (\overline{\chi^{j+1}_{\mathsf{R}}} - \overline{\chi^j_{\mathsf{R}}})\chi^j_{\mathsf{L}} \right) .
\end{aligned}$$

- Mass scales:

$$\begin{aligned}
m_\star &:= \Delta r \Delta\varphi \sqrt{|g_{55} g_{66}|} V_5 \frac{N+1}{A} \frac{1}{\Delta r} = \frac{2\pi(N+1)L}{NA}, \\
m &:= \Delta r \Delta\varphi \sqrt{|g_{55} g_{66}|} V_6 \frac{N+1}{A} \frac{1}{\Delta\varphi} = \frac{(N+1)L}{A\sqrt{1 - eL^2}} .
\end{aligned}$$

- Fermion mass ratio = graviton mass ratio:

$$\frac{m_\star^2}{m^2} = \frac{(2\pi)^2}{N^2}(1 - eL^2).$$

- Mass matrix:

$$M = m_\star \cdot \begin{bmatrix} & \chi_\mathsf{R}^0 & \chi_\mathsf{R}^1 & \chi_\mathsf{R}^2 & \cdots \\ \overline{\chi_\mathsf{L}^0} & 0 & 0 & 0 & \\ \overline{\chi_\mathsf{L}^1} & -1 & 1 & & \\ \overline{\chi_\mathsf{L}^2} & -1 & & 1 & \\ \vdots & \vdots & & & \ddots \end{bmatrix} + i \cdot m \cdot \begin{bmatrix} & \chi_\mathsf{R}^0 & \chi_\mathsf{R}^1 & \chi_\mathsf{R}^2 & \cdots \\ \overline{\chi_\mathsf{L}^0} & 0 & 0 & 0 & \\ \overline{\chi_\mathsf{L}^1} & 0 & -1 & 1 & \\ \overline{\chi_\mathsf{L}^2} & 0 & & -1 & 1 \\ \vdots & \vdots & & & \ddots \end{bmatrix}.$$

Diagonalisation by a bi-unitary transformation, left- and right-handed fields transform differently.
Mass eigenstates ψ^n:

$$\begin{aligned}
\overline{\chi_\mathsf{L}^0} &= \overline{\psi_\mathsf{L}^0}, \\
\overline{\chi_\mathsf{L}^j} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp(+2\pi i \cdot j \frac{n}{N}) \overline{\psi_\mathsf{L}^n}, \\
\chi_\mathsf{R}^0 &= \frac{1}{\sqrt{N+1}} \psi_\mathsf{R}^0 - \frac{N}{\sqrt{N(N+1)}} \psi_\mathsf{R}^N, \\
\chi_\mathsf{R}^j &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N-1} \exp(-2\pi i \cdot j \frac{n}{N}) \psi_\mathsf{R}^n + \frac{1}{\sqrt{N+1}} \psi_\mathsf{R}^0 + \frac{1}{\sqrt{N(N+1)}} \psi_\mathsf{R}^N.
\end{aligned}$$

- Action in terms of mass eigenstates ψ^j:

$$
\begin{aligned}
S \; = \; & \sum_{j=0}^{N} \int \mathrm{d}^4x \, \Big[\frac{1}{2} i \left(\overline{\psi^j} \gamma^\alpha \partial_\alpha \psi^j - \overline{\partial_\alpha \psi^j} \gamma^\alpha \psi^j \right) \\
& - \sum_{j=1}^{N-1} \int \mathrm{d}^4x \cdot \overline{\psi_{\mathsf{L}}^j} \psi_{\mathsf{R}}^j [m_\star + im(e^{-2\pi i \frac{n}{N}} - 1)] + \overline{\psi_{\mathsf{R}}^j} \psi_{\mathsf{L}}^j [m_\star - im(e^{2\pi i \frac{n}{N}} - 1)] \\
& - \sqrt{N+1}\, m_\star \left(\overline{\psi_{\mathsf{L}}^N} \psi_{\mathsf{R}}^N + \overline{\psi_{\mathsf{R}}^N} \psi_{\mathsf{L}}^N \right) \Big].
\end{aligned}
$$

  - Flat zero-mode ψ^0. Heavy fermion ψ^N with mass $m_\star\sqrt{N+1}$.
  - $N-1$ fermions $\psi^1, \ldots, \psi^{N-1}$ with complex masses, squared absolute values

$$
|m_\star + im(e^{-2\pi i \frac{n}{N}} - 1)|^2 = m_\star^2 + 4m^2 \sin^2(\frac{\pi n}{N}) + 2 m_\star m \sin(\frac{2\pi n}{N}).
$$

    Different from graviton case: additional "interference" term $2m_\star m \sin(\frac{2\pi n}{N})$ in the mass spectrum.
  - Matching with graviton mass spectrum ▶ slightly modified discretisation procedure for the angular direction:

$$
\partial_6 \chi \to \frac{\chi^{j+1} - \chi^j}{\Delta\varphi} \quad \Longrightarrow \quad \partial_6 \to i \cdot \frac{\chi^{j+\frac{1}{2}} - \chi^{j-\frac{1}{2}}}{\Delta\varphi}.
$$

    Mass eigenstates unchanged, but modified complex masses for the modes $\psi^1 \ldots \psi^{N-1}$.
    Squared absolute values:

$$
|m_\star + im(2\sin\frac{\pi n}{N})|^2 = m_\star^2 + 4m^2 \sin^2(\frac{\pi n}{N}).
$$

    Mass spectrum of the gravitons, no interference terms.

- Application: Small fermion masses
  - Discrete version of wave function suppression mechanism. Arkani-Hamed, Dimopoulos, Dvali, March-Russel
  - Particle standard model located on the centre site ▶ possible couplings to right-handed χ^0_{R}.
  - Yukawa coupling after SSB

$$\overline{L}\langle H\rangle\chi^0_{\mathsf{R}}.$$

    Left-handed lepton doublet $\overline{L}$, VEV of the Higgs doublet $\langle H\rangle$.
  - Large number N of lattice sites
    ▶ large mass $m_\star\sqrt{N+1}$ of heavy mode ψ^N
    ▶ ψ^N decouples.

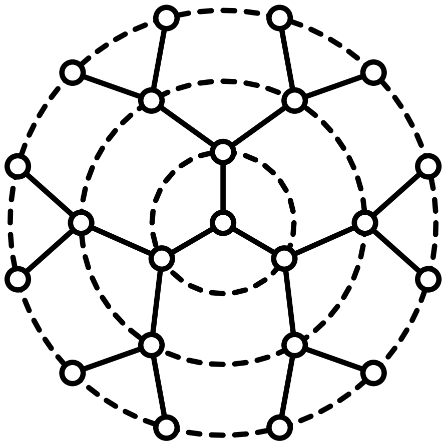
$$\chi^0_{\mathsf{R}} = \frac{1}{\sqrt{N+1}}\psi^0_{\mathsf{R}} - \frac{N}{\sqrt{N(N+1)}}\psi^N_{\mathsf{R}}.$$

  - Suppression of the SM neutrino mass:

$$\overline{L}\langle H\rangle\chi^0_{\mathsf{R}} \longrightarrow \frac{1}{\sqrt{N+1}}\nu_{\mathsf{L}}\langle H\rangle\psi^0_{\mathsf{R}}.$$

Refined scenario with warping

- Only discrete disk, no continuum description.



- k_{max} circles, label $k = 0 \ldots k_{\text{max}}$. Number of sites on k^{th} circle: $N_k = 3 \cdot 2^{k-1}$.
  Total number of sites: $N^{\text{tot}} = 3 \cdot 2^{k_{\text{max}}} - 2$ ▶ exponential growth of lattice sites.
- Each site is linked to 3 other sites, one ingoing, two outgoing.
- Scenario with "warping": local Planck scale on k^{th} circle M_k:

$$M_k = M_4 \cdot \epsilon^k, \quad \epsilon \sim 0,1.$$

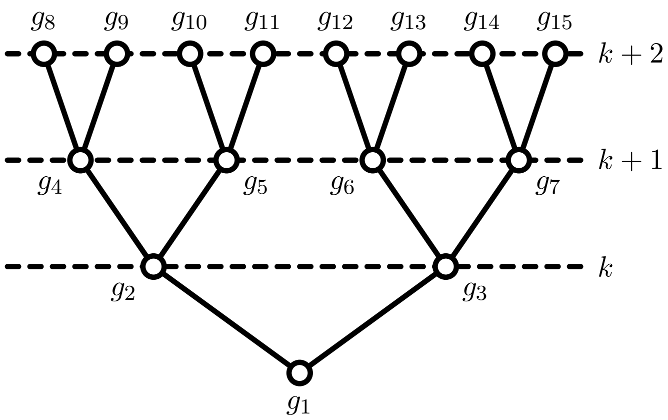
- Discretisation with constant mass scale in radial direction:

$$m_\star \approx \text{const.}$$

- Action:

$$S_{\text{mass}} = \sum_{k=1}^{k_{\text{max}}} \sum_{i,j} M_k^2 \int \mathrm{d}^4x \left[m_\star^2 \cdot (h_{\mu\nu}^{k,i} - h_{\mu\nu}^{k-1,j})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(h_{\alpha\beta}^{k,i} - h_{\alpha\beta}^{k-1,j}) \right].$$

- One branch:



- Example: First 7 gravitons $(h_{\mu\nu}^1, h_{\mu\nu}^2, \ldots, h_{\mu\nu}^7)$, mass matrix:

$$M_4^2 M_g^2 = M_4^2 m_*^2 \epsilon^{2k} \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1+2\epsilon^2 & 0 & -\epsilon^2 & -\epsilon^2 & 0 & 0 \\ -1 & 0 & 1+2\epsilon^2 & 0 & 0 & -\epsilon^2 & -\epsilon^2 \\ 0 & -\epsilon^2 & 0 & \epsilon^2 & 0 & 0 & 0 \\ 0 & -\epsilon^2 & 0 & 0 & \epsilon^2 & 0 & 0 \\ 0 & 0 & -\epsilon^2 & 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & -\epsilon^2 & 0 & 0 & 0 & \epsilon^2 \end{pmatrix}.$$

- General case (perturbation theory for $\epsilon \ll 1$):
  - j^{th} graviton on the k^{th} circle: $h_{\mu\nu}^{k,j}$, $\quad k = 0 \ldots k_{\text{max}}, \quad j = 1 \ldots N_k$.
  - Mass eigenstates of N^{tot} gravitons:

$$H^{k,j} \text{ with mass } m_\star^2 \lambda_{k,j}.$$

  - One zero mode, $\lambda_{0,0} = 0$:

$$H_{\mu\nu}^{0,0} = \frac{1}{\sqrt{N^{\text{tot}}}}(1, 1, \ldots, 1).$$

  - Two degenerate heavy modes, $\lambda_{1,1} = \lambda_{1,2} = \epsilon^2$:

$$H_{\mu\nu}^{1,j} = \frac{1}{\sqrt{3}}(0, 1, e^{\mathrm{i}2\pi j/3}, e^{\mathrm{i}4\pi j/3}, 0, \ldots, 0).$$

  - Heaviest mode, $\lambda_{1,3} = 4\epsilon^2$:

$$H_{\mu\nu}^{1,3} = \frac{1}{\sqrt{12}}(-3, 1, 1, 1, 0, \ldots, 0).$$

– For each $k = 2, 3, \ldots, k_{\text{max}}$:
$N_k - 1$ degenerate modes $\lambda_{k,j} = \epsilon^{2k}$:

$$H_{\mu\nu}^{k,j} = \frac{1}{\sqrt{N_k}}(\underbrace{0, \ldots, 0}_{N_{k-1}^{\text{tot}}}, \underbrace{1, e^{\mathrm{i}2\pi j/N_k}, e^{\mathrm{i}4\pi j/N_k}, \ldots, e^{\mathrm{i}2\pi j(N_k-1)/N_k}}_{N_k}, 0, \ldots, 0), \quad j = 1 \ldots N_k - 1.$$

Localised on the k^{th} circle.

– For each $k = 2, 3, \ldots, k_{\text{max}}$:
One slightly heavier mode, $\lambda_{k,N_k} \approx 2\epsilon^{2k}$:

$$H_{\mu\nu}^{k,N_k} = \frac{1}{\sqrt{N_k^{\text{tot}}}}(\underbrace{-1, \ldots, -1}_{N_{k-1}^{\text{tot}}}, \underbrace{1, \ldots, 1}_{N_k}, 0, \ldots, 0).$$

No support outside the k^{th} circle.

Summary and conclusions

- Continuum: Extra-dimensional disk, constant curvature.
- Special discretisation with only one site in the centre.
  - Massive 4D gravitons.
  - Mass spectrum has a gap: $(p = 1 \dots N-1)$

$$\begin{aligned} M_0^2 &= 0, \\ M_p^2 &= m_\star^2 + 4m^2 \sin^2\frac{\pi p}{N}. \end{aligned}$$

    One heavy mode: $M_N^2 = (N+1)m_\star^2$.
  - Mass scales $m_\star$, m determined by disk parameters. Mass ratio tunable:

$$\frac{m_\star^2}{m^2} = \frac{(2\pi)^2}{N^2}(1 - eL^2).$$

- 6D Dirac fermion on the discrete disk.
  - Massive 4D Dirac fermions, similar/equal spectrum as gravitons, same mass ratio $m_\star/m$.
  - Application: Small fermion masses by discrete wave function suppression mechanism.

- Refined discretisation.
  - Exponential growth of lattice sites in radial direction.
  - "Warping": local Planck scale depends on distance from centre.
  - Graviton mass spectrum and eigenstates. Localisation properties.
- Application:
  - Stronger correspondence between deconstruction models and discrete geometries.
  - Disk curvature ▶ more flexibility in choosing/tuning mass scales.
- Outlook/work in progress:
  - Graviton scattering might lead to strong coupling effects.
  - Determination of range of validity in effective theories of massive gravitons.