The C-Theorem for the Entanglement Entropy

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Entanglement or geometric entropy

- vacuum state $\rightarrow$ density matrix

$$|\Psi\rangle \rightarrow |\Psi\rangle\langle\Psi|$$

- spatial subset $A \rightarrow$ local density matrix ρ_A

$$|\Psi\rangle\langle\Psi| \rightarrow \rho_A = \mathrm{tr}_{-A}|\Psi\rangle\langle\Psi|$$

- entanglement entropy $\rightarrow S(A) = -\mathrm{tr}(\rho_A \log \rho_A)$

- alpha-entropies $\rightarrow S_\alpha(A) = \frac{1}{1-\alpha}\log \mathrm{tr}(\rho_A^\alpha)$

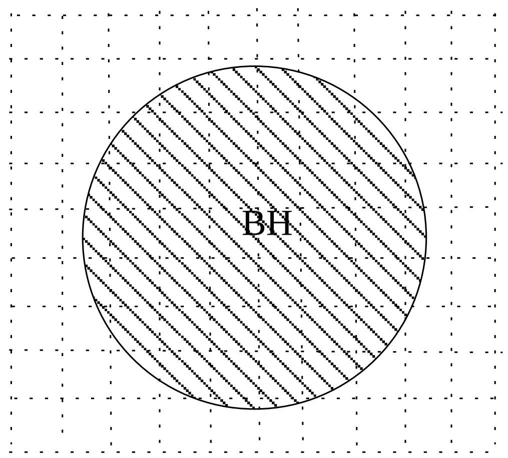
  $S(A) = \lim_{\alpha\rightarrow 1} S_\alpha(A)$

$S_\alpha(A)$ is proportional to the free energy in an Euclidean space with conical singularities (of angle $2\pi\alpha$) at the boundary of A

S(A) is a susceptibility under small conical singularities

Interests:

- Proposal to explain the entropy of BH.



- Condensed matter

  -Density matrix renormalization group

  -Quantum information theory

Quantum Field Theory

- S(A) as a non local variable with non-perturbative property:

  strong subadditivity (SSA)

  $S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$

  $S(A) = S(-A)$

- General features of QFT ($T_{\mu\nu}$; S)

  (Topological order, c-theorem...)

- Geometrical structure of divergences, which are proportional to local and extensive quantities depending on the boundary of A

$$S(A) = C_{d-1}\frac{R^{d-1}}{\epsilon^{d-1}} + C_{d-2}\frac{R^{d-2}}{\epsilon^{d-2}} + ... + C_0 \log(\epsilon) + S_f$$

(Ref:H.Casini and M.Huerta, hep-th/0606256)

$\rightarrow$ Consequence:

The mutual information is universal

$I(A, B) = S(A) + S(B) - S(A \cup B)$

$\rightarrow$ Comment:

The entropy must be divergent in order to have $I(A, B) \neq 0$ (or the space-time not well defined at sufficiently short scale)

(Ref: H.Casini, hep-th/0312238)

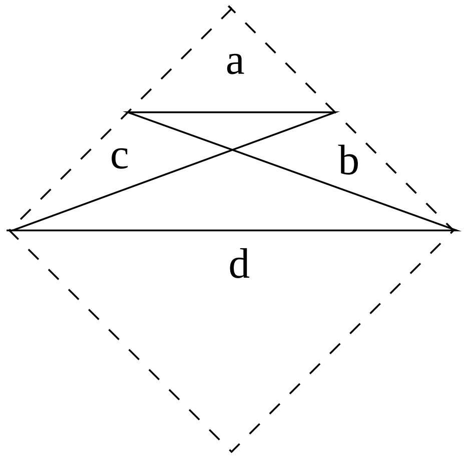
- What is the entropy contained in a given volume?

c-theorem in $1 + 1$ dimensions

$\rightarrow$ There is a function in the theories space in $(1 + 1)$ dimensions which is decreasing along the renormalization group trajectories and stationary at the fixed points where it takes a finite value proportional to the Virasoro central charge.

$\rightarrow$ There is a universal function c in $(1 + 1)$ dimensions defined for any theory which is dimensionless, decreasing under dilatations and takes finite values at fixed points.

Entropic c-theorem



Relativistic relation: $a.d = c.b$

$$SSA: \qquad S(c) + S(b) \geq S(a) + S(d)$$

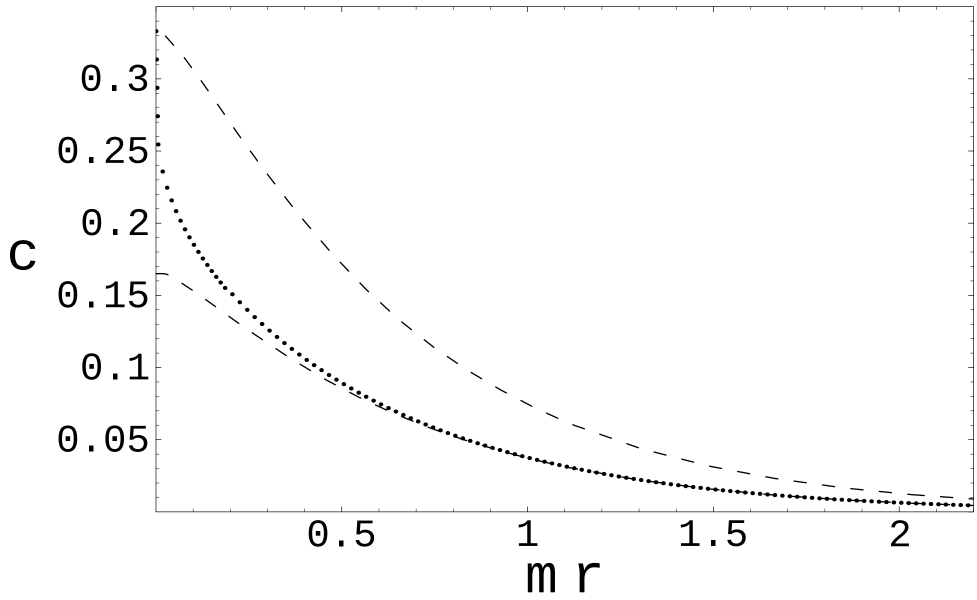
$$c = \lambda a, \quad d = \lambda b, \quad \lambda \geq 1$$

$$S(b) - S(a) \geq S(\lambda b) - S(\lambda a)$$

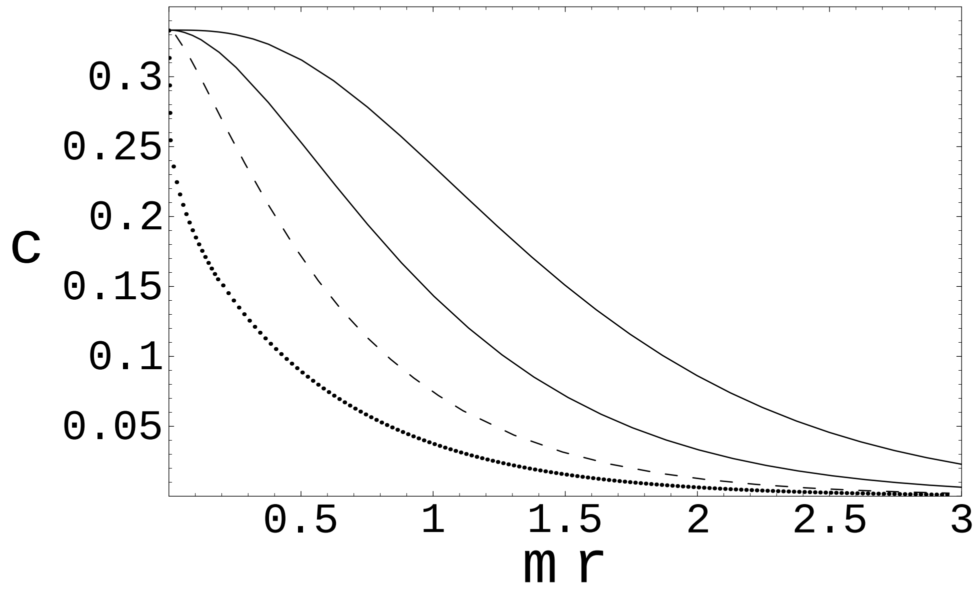
The difference of entropies $S(b) - S(a)$ is positive and decreasing under scaling. It is also dimensionless and universal. At fixed points takes the value $(C_V/3)\log(b/a)$. Thus, it is a c-function

→ Entropic c-function:

$$c(r) = r\frac{dS(r)}{dr}$$
$$c'(r) \leq 0$$



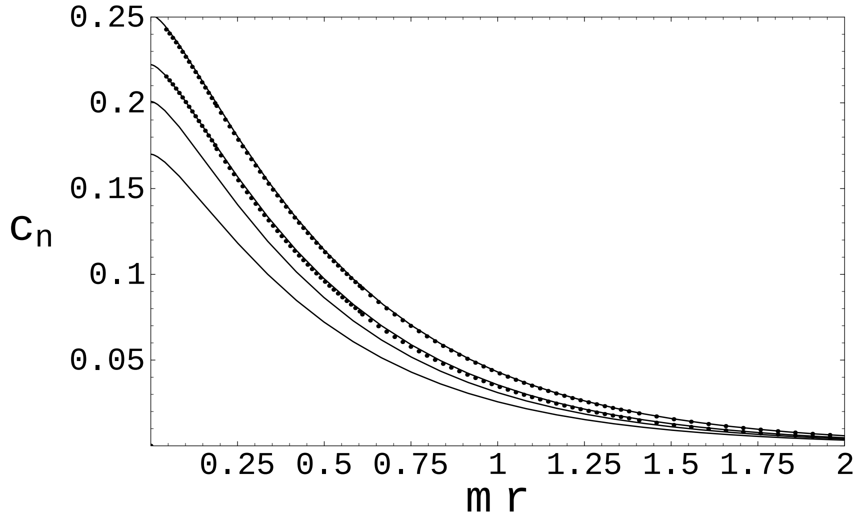
Calculations of $c(r)$ for free bosons and fermions in the lattice. From top to bottom: Dirac, real scalar and Majorana fields.



Comparison with Zamolodchikov c-functions. From top to bottom: Zamolodchikov c-functions for a real scalar and Dirac fields and entropic c-functions for Dirac and real scalar fields.

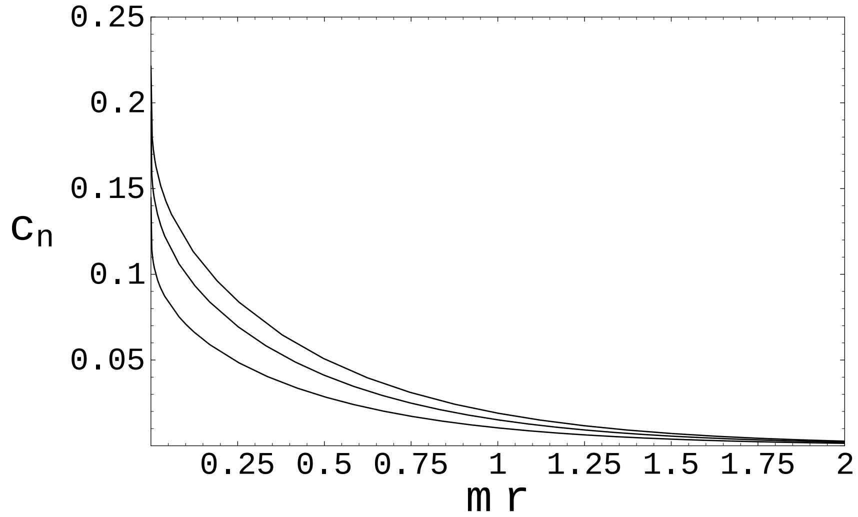
Analytical results for the alpha entropies

(Related to Painlevé V differential equations)



Dirac field c_n functions: $c_n = r\partial_r S_n(r)$ for $n = 2, 3, 5, 50$

(Ref: H.Casini, C.D.Fosco, M.Huerta, cond-mat/0505563)



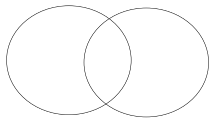
Real scalar field c_n functions: $c_n = r\partial_r S_n(r)$ for $n = 2, 3, 50$

(Ref: H.Casini,M.Huerta, cond-mat/0511014)

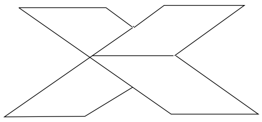
Is there a c-theorem for the α-entropies?

Is there a generalization of the entropic c-theorem to more dimensions?

- Not easy that intersections give the same type of object

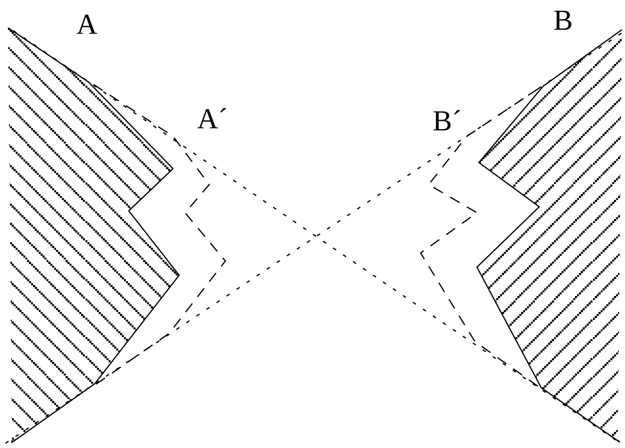


- No "diagonalizing" Cauchy surface for boosted sets



- Second order derivatives of the entropy coming from SSA for infinitesimally displaced sets is not enough to eliminate divergences

- Using $I(A,B)$?



$I(A,B)$ is universal, dimensionless and increasing with the size of A and B.

However $I(A,B)$ diverges logarithmically at critical points because there are terms in $S \sim \log \epsilon$ due to the vertices