CHAPLYGIN GAS
COSMOLOGY – unification of dark matter and dark energy

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Chaplygin gas

An exotic fluid with an equation of state

\[ p = -\frac{A}{\rho} \]

The first definite model for a dark matter/energy unification

The generalized Chaplygin gas

\[ p = -\frac{A}{\rho^\alpha} \quad 0 \leq \alpha \leq 1 \]

M.C. Bento, O. Bertolami, and A.A. Sen, PRD 66 (2002)

The term “quartessence” was coined to describe unified dark matter/dark energy models

• The Chaplygin gas model is equivalent to the (scalar) Dirac-Born-Infeld description of a D-brane

• String theory branes possess three features that are absent in the simple Nambu Goto $p$-dim membranes
  • (i) support an Abelian gauge field $A_\mu$ reflecting open strings with their ends stuck on the brane;
  • (ii) couple to the dilaton $\Phi$
  • (iii) couple to the (pull-back of) Kalb-Ramond field $B_{\mu\nu}$

D-branes and the Dirac-Born-Infeld (DBI) theory

\[ S_{\text{DBI}} = -\sqrt{A} \int d^{p+1}x \, e^{-\Phi} \sqrt{(-1)^p \det (g^{\text{ind}} + B)} \]

the induced metric ("pull back") of the bulk metric

\[ g_{\mu\nu}^{\text{ind}} = G_{ab} \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^b}{\partial x^\nu} \]

the antisymmetric tensor field

\[ B_{\mu\nu} = B_{\mu\nu} + 2\pi\alpha'F_{\mu\nu} \]
Choose the coordinates such that $X^\mu = x^\mu$ and let the $p+1$-th coordinate $X^{p+1} \equiv \theta$ be normal to the brane. From now on we set $p=3$. Then

\[
G_{\mu \nu} = g_{\mu \nu} \quad \text{for} \quad \mu = 0 \ldots 3
\]

\[
G_{\mu 4} = 0 \quad G_{44} = -1
\]
Scalar Born-Infeld theory

If we neglect the dilaton and the $B$-field, we find

$$S_{BI} = -\sqrt{A} \int d^4x \sqrt{-\det(g_{\text{ind}})} = \int d^4x L_{BI}$$

$$L_{BI} = -\sqrt{A} \sqrt{1 - \theta^2} \quad \theta^2 = g^{\mu\nu} \theta_{,\mu} \theta_{,\nu}$$

$$T_{\mu\nu} = \frac{\theta^2}{2} \frac{\partial L_{BI}}{\partial \theta^2} \theta_{,\mu} \theta_{,\nu} - L_{BI} g_{\mu\nu}$$

which is consistent with a perfect fluid description

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu} \quad \text{with} \quad u_\mu = \frac{\theta_{,\mu}}{\sqrt{\theta^2}}$$
\[ \rho = \frac{\sqrt{A}}{\sqrt{1 - \theta^2}}; \quad p = -\sqrt{A}\sqrt{1 - \theta^2} \]

and hence

\[ p = -\frac{A}{\rho} \]

In a homogeneous model the conservation equation yields the Chaplygin gas density as a function of the scale factor \( a \)

\[ p = \sqrt{A + \frac{B}{a^6}} \]

where \( B \) is an integration constant. The Chaplygin gas thus interpolates between dust (\( \rho \sim a^{-3} \)) at large redshifts and a cosmological constant (\( \rho \sim A^{\frac{1}{2}} \)) today and hence yields a correct homogeneous cosmology.
The inhomogeneous Chaplygin gas and structure formation

- The inhomogeneous Chaplygin gas based on a Zel'dovich type approximation has been proposed, and the picture has emerged that on caustics, where the density is high, the fluid behaves as cold dark matter, whereas in voids, \( w = p/\rho \) is driven to the lower bound \(-1\) producing acceleration as dark energy.
- Soon it has been shown that the naive Chaplygin gas model does not reproduce the mass power spectrum
  
  
  and the CMB
  
  D. Carturan and F. Finelli, PRD 68 (2003);
The physical reason is that although the adiabatic speed of sound

\[ c_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_s \]

is small until \( \alpha \sim 1 \), the accumulated comoving acoustic horizon

\[ d_s = \int dt \frac{c_s}{a} \approx H_0^{-1} a^{7/2} \]

reaches MPc scales by redshifts of twenty, frustrating the structure formation even into a mildly nonlinear regime

\[
\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\delta(1 + \delta) - \frac{4}{3} \frac{\dot{\delta}^2}{1 + \delta} - \frac{1 + \delta}{a^2} \nabla \left( \frac{c_s^2}{1 + \delta} \nabla \delta \right) = 0
\]

the density contrast described by a nonlinear evolution equation either grows as dust or undergoes damped oscillations depending on the initial conditions.

Evolution of the density contrast in the spherical model from \( \alpha_{eq} = 3 \cdot 10^{-4} \) for the comoving wavelength \( 1/k = 0.34 \) Mpc, \( \delta_k(\alpha_{eq}) = 0.004 \) (solid) \( \delta_k(\alpha_{eq}) = 0.005 \) (dashed).
The root of the structure formation problem is the last term, as may be understood if we solve the equation at linear order

$$\ddot{\delta} + 2H \dot{\delta} - \frac{3}{2} H^2 \delta - \frac{c_s^2}{a^2} \Delta \delta = 0$$

with the solution

$$\delta_k = a^{-1/4} J_{5/4}(d_s k)$$

behaving asymptotically as

$$\delta_k \sim a \quad \text{for} \quad d_s k \ll 1$$

$$\delta_k \sim \frac{\cos(d_s k)}{a^2} \quad \text{for} \quad d_s k \ll 1$$

Hence damped oscillations at the scales below $d_s$
The suggested ways out to overcome the acoustic barrier:

- A $k$-essence type-model where the Lagrangian has a local minimum
  

  Such a model is equivalent to the ghost condensate and hence shares its peculiarities
  
  A. Krause and S. P. Ng, IJMP A 21 (2006) 1091

- The generalized Chaplygin gas in a modified gravity approach, reminiscent of Cardassian models
  
• Another deformation of the Chaplygin gas

• If there are entropy perturbations such that the pressure perturbation $\delta p=0$ and with it the acoustic horizon $d_s=0$, no problem arises

The scenario with entropy perturbations, which is difficult to justify in the simple Chaplygin gas model, may be realized by introducing an extra degree of freedom, e.g., in terms of a quintessence-type scalar field $\varphi$. 
Nonadiabatic perturbations

Suppose that the matter Lagrangian depends on two degrees of freedom, e.g., a Born-Infeld scalar field $\theta$ and one additional scalar field $\phi$. In this case, instead of a simple barotropic form $p=p(\rho)$, the equation of state involves the entropy density (entropy per particle) $s$ and may be written in the parametric form

$p=p(\theta,\phi) \quad s=s(\theta,\phi)$

The corresponding perturbations

$$\delta p = \delta \rho \frac{\partial p}{\partial \rho} + \delta \phi \frac{\partial p}{\partial \phi}$$

$$\delta s = \delta \rho \frac{\partial s}{\partial \rho} + \delta \phi \frac{\partial s}{\partial \phi}$$
The speed of sound is the sum of two nonadiabatic terms

\[ c_s^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_s = \frac{\partial p}{\partial \rho} - \frac{\partial s}{\partial p} \left( \frac{\partial s}{\partial \phi} \right)^{-1} \frac{\partial p}{\partial \phi} \]

Thus, even for a nonzero \( \partial p/\partial \rho \) the speed of sound may vanish if the second term on the right-hand side cancels the first one. This cancellation will take place if in the course of an adiabatic expansion, the perturbation \( \delta \phi \) grows with \( \alpha \) in the same way as \( \delta \rho \). In this case, it is only a matter of adjusting the initial conditions of \( \delta \phi \) with \( \delta \rho \) to get \( c_s = 0 \).
Nonadiabatic perturbations in the DBI theory

The DBI theory which we started from possesses extra degrees of freedom in terms of the dilaton $\Phi$ and the tensor field $B$.

The full action contains the bulk terms in addition to the DBI Lagrangian

$$S = \int d^4 x \sqrt{-\det g} \left( \mathcal{L}_b + \mathcal{L}_{\text{DBI}} \right)$$

$$\mathcal{L}_b = \frac{1}{2\kappa^2} e^{-2\Phi} \left( -R - 4g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} + \frac{1}{12} B_{[\mu\nu,\sigma]} B^{[\mu\nu,\sigma]} \right)$$

$$\kappa^2 = 8\pi G$$
$$S_{\text{DBI}} = \int d^4 x \sqrt{-\text{det} \, g} \, \mathcal{L}_{\text{DBI}}$$

$$\mathcal{L}_{\text{DBI}} = -\sqrt{A} e^{-\Phi} \sqrt{(1 - \theta^2)(1 + \mathcal{B}^2) + \theta \mathcal{B}^2 \theta - (\mathcal{B} \ast \mathcal{B})^2}$$

Abbreviations:

$$\theta^2 = g_{\mu \nu} \partial_{,\mu} \theta \, \partial_{,\nu}$$

$$\mathcal{B}^2 = \frac{1}{2} \mathcal{B}_{\mu \nu} \mathcal{B}^{\mu \nu}$$

$$\mathcal{B} \ast \mathcal{B} = \frac{1}{8 \sqrt{-\text{det} \, g}} \varepsilon^{\mu \nu \rho \sigma} \mathcal{B}_{\mu \nu} \mathcal{B}^{\rho \sigma}$$

$$\theta \mathcal{B}^2 \theta = \theta_{,\mu} \mathcal{B}^{\mu} \mathcal{B}^{\nu \rho} \theta_{,\rho}$$
It is convenient to write everything in Einstein's frame using the transformation

\[ g_{\mu\nu} \rightarrow e^{2\Phi} g_{\mu\nu} \]

and simultaneously rescaling

\[ \theta_{,\mu} \rightarrow e^\Phi \theta_{,\mu} \quad B_{\mu\nu} \rightarrow e^{2\Phi} B_{\mu\nu} \]

which gives

\[ \mathcal{L}_{\text{DBI}} = -\sqrt{A} e^{3\Phi} \sqrt{(1 - \theta^2)(1 + B^2) + \theta B^2 \theta - (B * B)^2} \]

\[ \mathcal{L}_b = \frac{1}{2\kappa^2} \left( -R + 2g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} + \frac{1}{12} B_{[\mu\nu,\sigma]} B^{[\mu\nu,\sigma]} + V(\Phi, B) \right) \]

\[ V(\Phi, B) = \frac{1}{3} \Phi_{[\mu} B_{\nu\sigma]} B^{[\mu\nu,\sigma]} + \frac{1}{3} \Phi_{[\mu} B_{\nu\sigma]} \Phi^{[\mu} B^{\nu\sigma]} \]
The two parts of the energy–momentum tensor

\[ T_{\mu\nu} = T_{\mu\nu}^b + T_{\mu\nu}^{DBI} \]

are not in the form of a perfect fluid. To define \( \rho \) and \( p \), we make the decomposition

\[ T_{\mu\nu} = \rho u_\mu u_\nu - ph_{\mu\nu} + \Pi_{\mu\nu}; \]

\[ h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \]

\( \Pi_{\mu\nu} \) is a traceless anisotropic stress orthogonal to \( u_\mu u_\nu \) and \( h_{\mu\nu} \). Hence

\[ \rho = T_{\mu\nu} u_\mu u_\nu; \quad p = -\frac{1}{3} T_{\mu\nu} h_{\mu\nu} \]
Implications for linear perturbations

A convenient choice is the temporal (synchronous) gauge

$$ds^2 = dt^2 - a^2 (\delta_{ij} - f_{ij}) dx^i dx^j$$

First-order perturbations

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$$

$$\delta_\Phi = \Phi - \bar{\Phi}$$

$$\delta_\theta = \theta - \bar{\theta}$$

Homogeneity and isotropy imply

$$\bar{\Phi}_{,i} = 0$$

$$\bar{\theta}_{,i} = 0$$

$$\bar{B}_{\mu\nu} = 0$$

Hence, $\Phi_{,i}$, $\theta_{,i}$ and $(\bar{B}_{\mu\nu})^2$ count as first order
With these assumptions we may neglect higher-order terms in $T_{\mu\nu}$ and we find simple expressions for the density and the pressure

$$\rho \equiv u^\mu u^\nu T^{\text{DBI}}_{\mu\nu} = \sqrt{A} e^{3\Phi} \sqrt{\frac{1 + B^2}{1 - \theta^2}}$$

$$p \equiv -\frac{1}{3} h^{\mu\nu} T^{\text{DBI}}_{\mu\nu} = -\frac{A e^{6\Phi}}{\rho} \left( 1 + \frac{1}{3} B^2 \right)$$

associated with the DBI part $T^{\text{DBI}}_{\mu\nu}$ of the energy momentum
The pressure perturbation is given by

$$\delta p = -\bar{p}\left(\frac{\delta \rho}{\bar{\rho}} + 6 \delta \Phi - \frac{1}{3} B^2\right)$$

and hence, the nonadiabatic perturbation cancellation scenario is realized by

$$\frac{\delta \rho}{\bar{\rho}} + 6 \delta \Phi - \frac{1}{3} B^2 = 0$$

as an initial condition outside the causal horizon

$$d_c = \int dt/\alpha \approx H_0^{-1} \alpha^{1/2}$$
The dilaton

Retaining only the dominant terms, the dilaton perturbation satisfies

$$\ddot{\delta \Phi} + 3H \dot{\delta \Phi} - \frac{1}{a^2} \Delta \delta \Phi = 0$$

with the solution in \( k \)-space

$$\delta \Phi_k = a^{-3/4} J_{3/2} (d_c k)$$

Then, once the perturbations enter the causal horizon \( d_c \) (but are still outside the acoustic horizon \( d_s \)), \( \delta \Phi \) undergoes rapid damped oscillations, so that the nonadiabatic perturbation associated with \( \Phi \) is destroyed. This means that the nonadiabatic perturbations are not automatically preserved except at long, i.e., superhorizon, wavelengths where the simple Chaplygin gas has no problem anyway.
The Kalb-Ramond field

The temporal gauge ansatz

\[ \mathcal{B}_{0i} = 0 \quad \mathcal{B}_{ij} = \varepsilon_{ijk} B^k \quad \mathcal{B}^2 = \frac{B^i B^j}{a^4} \]

Retaining the dominant terms, the field equation takes the form

\[ \frac{d}{dt} \left( \frac{\dot{B}^i}{a} \right) - \frac{B_{,ji}^j}{a^3} + \frac{2\kappa^2 A}{\bar{\rho}} \frac{B^i}{a} = 0 \]
The last term is of the order $c_s^2 \mathcal{H}^2$ compared with the first term of the order $\mathcal{H}^2$ and may be neglected. The equation is further simplified to

$$\frac{d}{dt} \left( \frac{\dot{B}^i}{a} \right) - \frac{B^j_{,ji}}{a^3} = 0$$

If we make the decomposition

$$B^i = B^i_\perp + B^i_\parallel$$

$$\partial_i B^i_\perp = 0$$

the key point becomes evident: whereas the longitudinal part $B^i_\parallel$ suffers the same problem as $\delta \Phi$, the transverse part $B^i_\perp$ does not experience spatial gradients.
The longitudinal equation (in k-space)

\[
\frac{d}{dt} \left( \frac{\dot{B}_\parallel (k)}{a} \right) - \frac{k^2}{a^3} B_\parallel (k) = 0
\]

has the solution

\[
B_\parallel (k) = a^{5/4} J_{5/2}(d_c k)
\]

Hence, the longitudinal term $B_\parallel^2/a^4$ oscillates with an amplitude decreasing as $a^{-2}$.
The transverse part satisfies

\[
\frac{d}{dt} \left( \frac{\dot{B}_\perp}{a} \right) = 0 \quad \rightarrow \quad \frac{B^i \dot{B}^i}{a^4} = \frac{\text{const}}{H_0^2} a
\]

Hence, the transverse term grows linearly with the scale factor \( a \). Owing to this we can arrange nonadiabatic perturbations such that

\[
\frac{B^i \dot{B}^i}{a^4} - 3\delta = 0 \quad \rightarrow \quad \delta p = 0
\]
We have achieved the cancellation of nonadiabatic perturbations with the assurance that this will hold independent of scale until $a \sim 1$. With vanishing $c_s^2$, the spatial gradient term is absent, and the density contrast satisfies

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\delta = 0; \quad \delta > 0$$

with the growing mode solution

$$\delta \sim a$$

which is our main result: the growing mode overdensities here do not display the damped oscillations of the simple Chaplygin gas below $\delta$, but grow as dust. We remark that it matters little that this applies only for $\delta > 0$ since the Zel'dovich approximation implies that 92% ends up in overdense regions.
Conclusions

• The estimate presented here can be taken as a starting point for investigating questions beyond linear theory, in the complete general-relativistic perturbation analysis including the electric-type field $B_{0i}$.

• One might hope that the acoustic horizon does resurrect at very small scales to provide the constant density cores seen in galaxies dominated by dark matter.

• The "magnetic" nature of the Kalb-Ramond field could generate rotation - it is pertinent to note that the simple, nonrotating, self-gravitating Chaplygin gas has a scaling solution $\rho \sim r^{2/3}$ far different from the $\rho \sim r^{2/3}$ wanted for flat rotation curves.

• Adding new degrees of freedom to some extent spoils the simplicity of quartessence unification.

• Ultimately, the model must be confronted with large-scale structure and the CMB.
The DBI Lagrangian is invariant under the Kalb-Ramond gauge transformations

\[ B_{\mu\nu} \rightarrow B_{\mu\nu} + \xi_{\mu,\nu} - \xi_{\nu,\mu} \]
\[ A_\mu \rightarrow A_\mu + \xi_\mu \]
\[ \xi_\mu \rightarrow \xi_\mu + \phi,\mu \]
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