A frame-invariant approach to Scalar-Tensor Cosmology

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DESY Theory

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Outline

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Scalar-Tensor Theories

- $g_{\mu\nu} \longrightarrow (g_{\mu\nu}, \varphi)$

- Brans-Dicke Theory :
  $G \longrightarrow \varphi$

- Scalar fields in Cosmology:
  General Relativity $\longrightarrow$ *ST*

EF and JF Scalar-Tensor Theories

- Einstein frame action

$$S_G = \frac{M_*^2}{2}\int d^4x\,\sqrt{-g}\,[R(g) - 2\,g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - 4V(\varphi)] + S_M\left[e^{-2b(\varphi)}g_{\mu\nu},\,\phi,\,\psi\right]$$

$$\tilde{g}_{\mu\nu} = e^{-2b(\varphi)}g_{\mu\nu}$$

- Jordan frame action

$$S_G = \frac{M_*^2}{2}\int d^4x\,\sqrt{-\tilde{g}}\,e^{2b(\varphi)}\left[R(\tilde{g}) - 2\,\tilde{g}^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi\,(1 - 3\,\alpha^2) - 4\tilde{V}(\varphi)\right] + S_M\left[\tilde{g}_{\mu\nu},\,\tilde{\phi},\,\tilde{\psi}\right]\;; \qquad \alpha \equiv \frac{db}{d\varphi}$$

The standard prescription

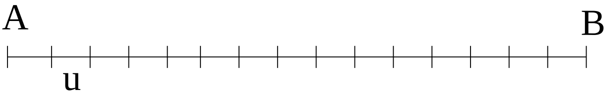
- Example of a non frame-invariant result:
  freeze-out $\rightarrow H \gtrsim a\Gamma$

- the standard prescription: solve the cosmological equations in the Einstein frame, translate the results in the Jordan frame and, finally, compare such a Jordan frame results with the experiments.

The Weyl rescaling

$$\tilde{g}_{\mu\nu} = e^{-2b(\varphi)} g_{\mu\nu}$$

The Dicke's argument



$L \equiv \bar{AB}$; $\quad u \equiv$ unit of measure $\Longrightarrow l = \frac{L}{u}$

$$\begin{aligned} u &\longrightarrow \lambda^{-1/2} u \\ l &\longrightarrow \lambda^{1/2} l \end{aligned}$$

$$ds \equiv (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$$

$$\begin{aligned} ds &\longrightarrow \lambda^{1/2} ds \\ g_{\mu\nu} &\longrightarrow \lambda g_{\mu\nu} \end{aligned}$$

$$\lambda(x^\mu) = e^{-2b[\varphi(x^\mu)]}$$

Transformations under a change of units

$$
\begin{aligned}
l &\longrightarrow e^{-b(\varphi)} l \\
\vec{x} &\longrightarrow \vec{x} \\
\tau &\longrightarrow \tau \\
m &\longrightarrow e^{b(\varphi)} m \\
\phi &\longrightarrow e^{b(\varphi)} \phi \\
\psi &\longrightarrow e^{\frac{3}{2} b(\varphi)} \psi \\
A_\mu &\longrightarrow A_\mu \\
\Gamma &\longrightarrow e^{b(\varphi)} \Gamma \\
\sigma &\longrightarrow e^{-2b(\varphi)} \sigma \\
&\vdots
\end{aligned}
$$

The reference length l_R

$$l_R \longrightarrow e^{-b(\varphi)} l_R$$

$$\begin{aligned}
\bar{m} &\equiv l_R m \\
\bar{\phi} &\equiv l_R \phi \\
\bar{\psi} &\longrightarrow l_R^{\frac{3}{2}} \psi \\
&\vdots \\
\bar{\Gamma} &\equiv l_R \Gamma \\
\bar{\sigma} &\equiv l_R^{-2} \sigma \\
h_{\mu\nu} &\equiv l_R^{-2} g_{\mu\nu}
\end{aligned}$$

Frame-invariant action

$$S = S_G[h_{\mu\nu}, \varphi] + S_M[h_{\mu\nu} e^{-2b[\varphi]}, \bar{\phi}, \bar{\psi}, \ldots ; \bar{\lambda}_n]$$

$$S_G = \kappa \int d^4x \sqrt{-h} \left[R(h) - 2\, h^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - 4\bar{V}(\varphi) \right]$$

Some comments

$$1 + z \equiv \frac{\lambda_o}{\lambda_e}$$

$$\lambda_o \longrightarrow e^{-b(\varphi_o)}\lambda_o$$

$$\lambda_e \longrightarrow e^{-b(\varphi_e)}\lambda_e$$

$\Rightarrow \frac{\lambda_o}{\lambda_e}$ is not frame-invariant

$$1 + z \equiv \frac{\lambda_o}{\lambda_e}\frac{l_R^e}{l_R^o}$$

- To work with dimensionless quantities is not a sufficient condition to have a frame-invariant formalism

- In this language to choose a frame corresponds to choose a system of units

$$S_G = \kappa \int d^4x \sqrt{-h} \left[R(h) - 2\, h^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - 4\bar{V}(\varphi) \right]$$

- $l_R \equiv const \Longrightarrow EF$

- $l_R \equiv const \times e^{-b(\varphi)} \Longrightarrow JF$

Energy-Momentum Tensor

$$P_\mu = m g_{u\nu} \frac{dx^\nu}{\sqrt{-ds^2}} = \tilde{m} \tilde{g_{\mu\nu}} \frac{dx^\nu}{\sqrt{-d\tilde{s}^2}}$$

(x^i, P_j) frame-invariant phase-space

$$f(x^i, P_j, \tau) dx^1 dx^2 dx^3 dP_1 dP_2 dP_3 = dN$$

$$\bar{T}^\mu_\nu \equiv l_R^4 T^\mu_\nu = g_s \int \frac{d^3P}{(2\pi)^3} (-g)^{-1/2} \frac{P^\mu P_\nu}{P^0} f(x^i, P_j, \tau)$$

Background equations

- $$\left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 - \frac{2}{3}\left(\frac{1}{2}\dot{\varphi}^2 + \bar{a}^2\bar{V}\right) = \frac{1}{6\kappa}\bar{\rho}\bar{a}^2$$
  $$\frac{\ddot{\bar{a}}}{\bar{a}} + \frac{1}{3}\left(\dot{\varphi}^2 - 4\bar{a}^2\bar{V}\right) = -\frac{1}{12\kappa}\bar{\rho}\bar{a}^2(1-3w)$$

- $$\dot{\bar{\rho}} + 3\,\bar{\rho}\,(1+w)\,\dot{\bar{a}}/\bar{a} = -\alpha\,\dot{\varphi}\,\bar{\rho}\,(1-3w)$$

- $$\ddot{\varphi} + 2\,\frac{\dot{\bar{a}}}{\bar{a}}\,\dot{\varphi} + \bar{a}^2\,\frac{\partial\bar{V}}{\partial\varphi} = \frac{\alpha}{4k}\,\bar{a}^2\,\bar{\rho}\,(1-3w)$$

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$$\dot{\bar{\rho}} + 3\,\bar{\rho}\,(1+w)\,\dot{\bar{a}}/\bar{a} = -\alpha\,\dot{\varphi}\,\bar{\rho}\,(1-3w)$$

$$\bar{\rho} = (I_R^E)^4 \rho_E = (I_R^J)^4 \rho_J\,; \qquad \bar{a} = a_E/I_R^E = a_J/I_R^J$$

- $I_R^E \equiv const$

$$\dot{\bar{\rho}}_E + 3\,\bar{\rho}_E\,(1+w)\,\dot{\bar{a}}_E/\bar{a}_E = -\alpha\,\dot{\varphi}\,\bar{\rho}_E\,(1-3w)$$

- $I_R^J \equiv const \times e^{-b(\varphi)}$

$$\dot{\bar{\rho}}_J + 3\,\bar{\rho}_J\,(1+w)\,\dot{\bar{a}}_J/\bar{a}_J = 0$$

Perturbations

$$ds^2 = a^2(\tau)\left[-(1+2\,\Psi)d\tau^2 + (1-2\,\Phi)\delta_{ij}dx^i dx^j\right]$$

$$ds^2 \longrightarrow d\tilde{s}^2 \;=\; e^{-2b(\varphi)}ds^2$$

$$b(\vec{x},\tau) = \bar{b}(\tau) + \delta b(\vec{x},\tau)$$

$$ds^2 = a^2(\tau)e^{-2\bar{b}}\left[-(1+2\,\Psi-2\,\delta b)d\tau^2 + (1-2\,\Phi-2\,\delta b)\delta_{ij}dx^i dx^j\right]$$

$$\Psi \longrightarrow \Psi - \delta b$$

$$\Phi \longrightarrow \Phi + \delta b$$

$$\bar{a} \equiv \frac{a}{l_R}$$

$$\bar{\Psi} \equiv \Psi - \frac{\delta l_R}{l_R}$$

$$\bar{\Phi} \equiv \Phi + \frac{\delta l_R}{l_R}$$

$$dh^2 = \bar{a}^2(\tau)\left[-(1+2\,\bar{\Psi})d\tau^2 + (1-2\,\bar{\Phi})\delta_{ij}dx^i dx^j\right]$$

Frame-invariant freeze-out

$$\frac{\partial f_\chi}{\partial \tau} + \frac{dx^i_\chi}{d\tau}\frac{\partial f_\chi}{\partial x^i_\chi} + \frac{dP^\chi_j}{d\tau}\frac{\partial f_\chi}{\partial P^\chi_j} = \left[\frac{df_\chi}{d\tau}\right]_C$$

$\chi \longleftrightarrow a + b$ $\quad\quad$ $\left(x \equiv \frac{m_\chi}{T}\right)$

$$\frac{d\, n^\chi_c}{d \log x} = -\frac{\Gamma\, a}{H(1 + m'_\chi/m_\chi)}(n^\chi_c - n^{\chi 0}_c)$$

$$\Gamma\, a \gtrsim H(1 + m'_\chi/m_\chi)$$

Frame-invariant geodesics

$$P^0 \frac{dP^\mu}{d\tau} + \Gamma^\mu_{\lambda\sigma} P^\lambda P^\sigma = -m\, \partial_\sigma m\, g^{\sigma\mu}$$

$$P_i = (1 - \bar{\Psi}) q_i$$

$$P_0 = -(1 + \bar{\Psi}) \epsilon$$

$$\epsilon = \sqrt{q^2 + \bar{a}^2 \bar{m}^2} \qquad q_i \equiv q n_i$$

$$\dot{q} = q \frac{\partial}{\partial \tau} \bar{\Phi} - \epsilon\, n_i \frac{\partial}{\partial x^i} \bar{\Psi}$$

Conclusions

- EF and JF are related each other by a transformation of units
- We introduced a frame-invariant formalism and, as an example, we applied it to the geodesics motion and to the Boltzmann equation
- As in perturbation theory all the physical observables can be written in terms of gauge-invariant variables, in scalar-tensor theory all the physical observables can be written in terms of frame-invariant variables