1 Dirac equation

1.1 Relativistic wave equations. Dirac equation

1.2 Non-relativistic limit

1.3 Solutions of the Dirac equation. Particles and antiparticles

Let us consider a particle at rest $\vec{p} = 0$. In this case Dirac equation reduces to

$$i\hbar \frac{\partial \psi}{\partial t} = mc^2 \beta \psi = mc^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \psi. \tag{1}$$

It is immediate to see that the 4-dimensional vectors ("spinors") that follow are solution of this equation

$$\psi_r(x) = \omega_r e^{-i\epsilon_r \frac{mc^2}{\hbar} t}, \tag{2}$$

with

$$\omega_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \omega_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \omega_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \omega_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3}$$

and $\epsilon_r = +1$ for $r = 1, 2$ and $\epsilon_r = -1$ for $r = 3, 4$.

We see at once that Dirac equation has, exactly as the Klein-Gordon equation, negative energy solutions that we have to interpret. In order to obtain the solution for $\vec{p} \neq 0$ we can apply a boost. While it is more or less clear how the space time dependence appears

$$e^{-i\epsilon_r \frac{mc^2}{\hbar} t} \to e^{-i\epsilon_r px}, \tag{4}$$

the transformation properties of the spinors are less clear. We shall postpone this discussion (see below).

We have to face the problem of interpreting negative energy solutions. Dirac proposed that the vacuum state is such that all negative energy states are filled up. Since we are dealing with fermions, it is impossible for any such state to be further occupied. This is called the Dirac sea. If we add fermions they an only occupy a positive energy state. Further there is a gap $2m$ between the highest negative energy state and the lowest possible positive energy state.

If we inject energy into the system (por instance, via a virtual photon or a couple of real photons) it is possible to promote one of the negative energy states to positive energy. We have thus produced a "particle" with positive energy and a "hole" in the sea of negative energy, i.e. also positive energy (and opposite momentum too). This is an *antiparticle*.

We now understand that the solutions labelled u correpond to particles and the solutions labelled v, that have negative energy, correspond to antiparticles, or, rather, the absence of one such solutions is a physical antiparticle.

From now on, unless explicitly stated otherwise we shall set $c = 1$.

1.4 Clifford algebra

It is defined by the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{5}$$

In addition the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^2\alpha\gamma^\beta$ is also introduced. We shall see later its utility There are infinite representations of this algebra, but three of them are commonly used. Recall the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Dirac representation:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{7}$$

Chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}. \tag{8}$$

Majorana representation:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}. \tag{9}$$

In $d = 4$ there are 16 independent combinations of gamma matrices

$$\begin{aligned}
&I, \\
&\gamma^0, \gamma^1, \gamma^2, \gamma^3, \\
&\gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3, i\gamma^1\gamma^2, i\gamma^1\gamma^3, i\gamma^2\gamma^3, \\
&i\gamma^0\gamma^5, \gamma^1\gamma^5, \gamma^2\gamma^5, \gamma^3\gamma^5, \\
&\gamma^5.
\end{aligned}$$

All these matrices satisfy

$$
\begin{aligned}
(\Gamma^A)^2 &= I \\
Tr\Gamma^A &= 0 \\
\Gamma^A\Gamma^B &= \lambda\Gamma^C \\
\forall X,\ X &= \frac{1}{4}\sum_A Tr(X\Gamma^A)\Gamma^A \\
\forall A \neq 1, \exists B; \{\Gamma^A, \Gamma^B\} &= 0.
\end{aligned}
$$

Other important properties following directly from the definition of γ^5 and the anti-commutation relation defining Clifford algebra are:

$$\{\gamma^\mu, \gamma^5\} = 0, \tag{10}$$

$$(\gamma^0)^2 = I, \quad (\gamma^i)^2 = -I, \quad (\gamma^5)^2 = I, \tag{11}$$

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \quad (\gamma^5)^\dagger = \gamma^5, \quad \gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu, \tag{12}$$

Traces of Dirac matrices:

$$
\begin{aligned}
Tr(\not{A}) &= 0 \\
Tr(\not{A}_1\ \not{A}_2\ \not{A}_{2n+1}) &= 0 \\
Tr(\not{A}\ \not{B}) &= 4AB \\
Tr(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) &= 4(g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}) \\
Tr(\gamma^5) &= 0 \\
Tr(\gamma^\mu\gamma^5) &= Tr(\gamma^\mu\gamma^\nu\gamma^5) = Tr(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^5) = 0 \\
Tr(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^5) &= 4i\epsilon^{\mu\nu\alpha\beta}.
\end{aligned}
$$

with

$$\epsilon^{0123} = -1, \qquad \epsilon_{0123} = 1. \tag{13}$$

1.5 Covariance of the Dirac equation

We all know the transformation properties of vectors under Lorentz transformations

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu}x^\nu \Rightarrow \partial_\mu = \Lambda^\nu_{\ \mu}\partial'_\nu \tag{14}$$

$$\eta_{\mu\nu}\Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta} = \eta_{\alpha\beta}. \tag{15}$$

The 'spinors', i.e. the solutions of the Dirac equation are 4-dimensional objects, but how do they transform under the Lorentz group in order to preserve covariance?

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x), \tag{16}$$

where S has to be a unitary matrix $S^\dagger = S^{-1}$.

Taking as starting point the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0, \tag{17}$$

$$(i\gamma^\mu\Lambda^\nu_{\ \mu}\partial'_\nu - m)S^{-1}(\Lambda)\psi'(x') = 0 \tag{18}$$

multiplying by $S(\Lambda)$

$$\left(iS(\Lambda)\gamma^\mu\Lambda^\nu_{\ \mu}S^{-1}(\Lambda)\partial'_\nu - m\right)\psi'(x') = 0 \tag{19}$$

$$\Rightarrow S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu_{\ \mu} = \gamma^\nu. \tag{20}$$

or

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu_{\ \mu}\gamma^\mu. \tag{21}$$

Let us now consider infinitessimal transformations

$$\Lambda^\mu_\nu \simeq g^\mu_\nu + \delta w^\mu_\nu \qquad S(\Lambda) = I - \frac{i}{4}A^{\mu\nu}\delta w_{\mu\nu}. \tag{22}$$

$A_{\mu\nu}$ is antisymmetric (or rather, only the antisymmetric part matters). Then expanding the relation

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu_{\ \mu}\gamma^\mu. \tag{23}$$

we get at the first non-trivial order

$$\frac{i}{4}A^{\rho\sigma}\gamma^\mu\delta w_{\rho\sigma} - \frac{i}{4}\gamma^\mu A^{\alpha\beta}\delta w_{\alpha\beta} = \delta w^\mu_{\ \nu}\gamma^\nu. \tag{24}$$

We leave as an exercise to prove that

$$A^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{25}$$

is the only solution. The $\sigma^{\mu\nu}$ are the Poincaré generators acting on spin 1/2 fields.

The adjoint spinor $\bar{\psi} = \psi^\dagger\gamma^0$ transforms in the following way

$$\psi'(x') = \psi^\dagger(x)S^\dagger(\Lambda)\gamma^0 = \bar{\psi}(x)S^\dagger(\Lambda). \tag{26}$$

Thus $\bar{\psi}(x)\psi(x)$ is an scalar.

1.6 Spinor properties

In the reference frame where the particle is at rest, $k^\mu = (m, 0)$ we already know the solutions of the Dirac equation. Our objective now is to obtain the solutions in a general reference frame. In order to do so, we have to first rotate them and then boost them.

A Lorentz transformation acting on a spinor is given by the expression (16), with $S(\Lambda)$ given by

$$S(\Lambda) = \exp -\frac{i}{4} w_{\mu\nu} \sigma^{\mu\nu}. \tag{27}$$

If we move to the Dirac representation, the rotations generators are

$$\sigma^{ij} = \epsilon^{ij}_{\ \ k} \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \tag{28}$$

and

$$\frac{1}{4} w_{ij} \sigma^{ij} = \frac{1}{4} \theta \hat{n}^k \epsilon_{kij} \sigma^{ij} = \theta n^i J_i, \tag{29}$$

with

$$J^i = \frac{1}{4} \epsilon^{ijk} \sigma_{jk}. \tag{30}$$

Eq (29), together with eq. 28), is a much more familiar expression for the action of the rotation group on the spinorial degrees of freedom Using these expressions we can perform a rotation on a spinor (originally pointing in the z direction) by means of the Euler angles

$$R(\theta, \phi) = e^{-i\phi J^3} e^{-i\theta J^2}, \tag{31}$$

where e.g.

$$e^{-i\theta J^2} = \cos \frac{\theta}{2} I + i \sin \frac{\theta}{2} \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}. \tag{32}$$

The boosts generators, in the Dirac representation, are given by

$$\sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{33}$$

Therefore, a boost in the direction of $\hat{k}$ is performed by means of the following operator

$$B(k) = e^{-i\psi \hat{k} N}, \tag{34}$$

with $N^i = \sigma^{0i}/2$ and $\psi = |k|/E$.

$$B(k) = \cosh \frac{\psi}{2} I + \frac{1}{|k|} \sinh \frac{\psi}{2} \begin{pmatrix} 0 & \hat{k} \cdot \sigma \\ \hat{k} \cdot \sigma & 0 \end{pmatrix} \tag{35}$$

Now we are in a position to apply succesively these transformations and obtain the general solutions for the spinors. Below we give the transformed spinors after a boost of 3-momentum $\vec{k}$

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{k_3}{E+m} \\ \frac{k_1+ik_2}{E+m} \end{pmatrix} \qquad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{k_1-ik_2}{E+m} \\ \frac{-k_3}{E+m} \end{pmatrix}, \tag{36}$$

and

$$v_1 = N \begin{pmatrix} \frac{k_3}{|E|+m} \\ \frac{k_1+ik_2}{|E|+m} \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = N \begin{pmatrix} \frac{k_1-ik_2}{|E|+m} \\ \frac{-k_3}{|E|+m} \\ 1 \\ 0 \end{pmatrix}. \tag{37}$$

N is a normalization constant. Note that the negative energy components acquire non-zero values.

They are solution of the equations (in momentum space)

$$(\not{k} - m)u_r = 0 \qquad (-\not{k} - m)v_r = 0. \tag{38}$$

The solutions associated to u_r correspond to particles of 3-momentum $\vec{k}$ and energy $E = \sqrt{(\vec{k})^2 + m^2}$, while the ones associated to the v_r correspond to states of 3-momentum $-\vec{k}$ and energy $E = -\sqrt{(\vec{k})^2 + m^2}$. The *absence* of a state with momentum $-\vec{k}$ and energy $-E$ in the Dirac sea is *equivalent* to the *presence* of a state with momentum $\vec{k}$ and energy E; the antiparticles.

The following properties hold for the spinors (this selects the normalization constant N)

$$\bar{u}(\vec{k},\sigma)u(\vec{k},\sigma') = -\bar{v}(\vec{k},\sigma)v(\vec{k},\sigma') = 2m\delta_{\sigma,\sigma'}, \tag{39}$$

$$\sum_\sigma u(\vec{k},\sigma)\bar{u}(\vec{k},\sigma) = \not{k} + m, \qquad \sum_\sigma v(\vec{k},\sigma)\bar{v}(\vec{k},\sigma) = \not{k} - m. \tag{40}$$

1.7 P, C and chirality

We will now discuss two important discrete symmetries that are best understood by the way they act on fermions. Let us now consider the following projection operators

$$P\pm = \frac{1}{2}(1 \pm \gamma_5) \tag{41}$$

and take, for the time being, $m = 0$.

We define the chirally projected fermions $\psi_{L,R}$

$$\psi_L = P_- \psi,\ \psi_R = P_+ \psi \tag{42}$$

so

$$\bar{\psi}_L = \bar{\psi} P_+,\ \bar{\psi}_R = \bar{\psi} P_+ \tag{43}$$

The L (left) and R (right) states contain half the Dirac fermion degrees of freedom. They are in fact helicity eigenstates for $m = 0$. To see that we write the Dirac equation in the $\vec{k} = \hat{z}$ frame; then $k^1 = k^2 = 0$, $k^0 = k^3$, so Dirac implies

$$k(\gamma^0 - \gamma^3)\psi = 0 \Rightarrow \gamma^0 \psi = \gamma^3 \psi. \tag{44}$$

The angular momentum operator (a generator of the Poincaré group) acting on spin 1/2 particles is

$$J^i = \frac{1}{4}\epsilon^{ijk}\sigma^{jk},\ \sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k] \tag{45}$$

so

$$J^3 = \frac{1}{2}\sigma^{12} = \frac{i}{2}\gamma^1\gamma^2. \tag{46}$$

Then

$$J^3\psi_L = \frac{1}{2} i\gamma^1\gamma^2 \frac{1-\gamma_5}{2}\psi = \frac{1}{2}\frac{1-\gamma_5}{2} i\gamma^0\gamma^0\gamma^1\gamma^2\psi = \frac{1}{2}\frac{1-\gamma_5}{2}\gamma^5\psi = -\frac{1}{2}\psi_L \tag{47}$$

and $J^3\psi_R = +\frac{1}{2}\psi_R$. The chirally projected fermions are, in the massless case, helicity eigenstates. Helicity is defined as

$$\frac{\vec{J}\vec{p}}{|p|}. \tag{48}$$

If $m \neq 0$, chirality and helicity do not coincide. Chirality is not a conserved quantum number for massive particles.

Since helicity involves a pseudovector, its sign must change under a parity transformation. Let us see how parity can be implemented at the operator level on fermions. Let us consider the free Dirac equation (in momentum space)

$$(\not{p} - m)\psi = (p^0\gamma^0 - p^i\gamma^i - m)\psi = 0. \tag{49}$$

Under a parity transformation: $p^0 \to p^0, p^i \to -p^i$

$$(p^0\gamma^0 + p^i\gamma^i - m)\psi' = 0. \tag{50}$$

Clearly $\psi' = e^{i\varphi}\gamma^0\psi$, and $\varphi = 0, \pi$. If the Dirac equation contains a coupling to an external vector field, this changes accordingly under parity: $A^0 \to A^0$, $A^i \to -A^i$.

Note that for *massive* particles, chirality is not a good quantum number. The mass term turns left landed into right handed. On the other hand, gauge interactions (vector or axial-vector) always preserve helicity, while scalar and pseudoscalar interactions do not.

Next we turn to charge conjugation C. Its meaning is best seen by considering the coupling of fermions to an external gauge field. According to the minimal coupling principle, the Dirac equation reads

$$i \not\partial\psi - e \not A\psi - m\psi = 0 \tag{51}$$

(We shall often use the *covariant derivative* $D_\mu = \partial_\mu + ieA_\mu$.) Obviously the equation describing the coupling to antiparticles should be

$$i \not\partial\psi_c + e \not A\psi_c - m\psi_c = 0 \tag{52}$$

Let us find the relation between ψ and ψ_c. We complex conjugate the first equation

$$-i\gamma_\mu^*\partial^\mu\psi^* - e\gamma_\mu^* A^\mu\psi^* - m\psi^* = 0 \tag{53}$$

Let us now multiply by a matrix C such that

$$C\gamma_\mu^* C = -\gamma_\mu \tag{54}$$

and such that $C^2 = 1, C^\dagger C = 1$. Elementary manipulations allow us to show that

$$-iC\gamma_\mu^* C\partial^\mu C\psi^* - eC\gamma_\mu^* CA^\mu C\psi^* - mC\psi^* = 0 \tag{55}$$

so, clearly,

$$\psi_c = C\psi^* \tag{56}$$

It can be seen that for Dirac fermions $C = i\gamma^2$.