5 Propagators and Green functions

5.1 The causality issue

Quantum field theory is buit over the same postulates as quantum mechanics, but with fields and conjugate canonical momenta replacing positions and ordinary momenta. A key ingredient, a priori absent in QM, is causality; i.e. the statement that information cannot propagate faster than light.

A good way analyzing this issue is by considering the Green function or propagator $G(\vec{x}, \vec{x}'; t)$. This is defined as the probability amplitude fot a particle that is initially localized at point $\vec{x}$ to be at point $\vec{x}'$ after a time t has elapsed

$$G(\vec{x}, \vec{x}'; t) = \langle \vec{x}' | e^{-iHt} | \vec{x} \rangle. \tag{199}$$

H is assumed to be time-independent (this is almost invariably the case in field theory) and, in fact we shall assume it is the free hamiltonian. This is also the relevant case for field theory since perturbation theory is the analytical tool that is commonly used. Between interactions particles are assumed to propagate freely.

It is a well known reault in QM that

$$G_0(\vec{x}, \vec{x}'; t) = \langle \vec{x}' | e^{-iH_0 t} | \vec{x} \rangle \tag{200}$$

$$= \int d^3p \langle \vec{x}' | e^{-i\frac{P^2}{2m}t} | \vec{p} \rangle \langle \vec{p} | \vec{x} \rangle = \left(\frac{m}{2\pi i t} \right)^{\frac{3}{2}} e^{im\frac{|\vec{x}'-\vec{x}|^2}{2t}} \tag{201}$$

From this expression it is quite apparent that the propagator is non-zero bewtween any two space-time points regardless whether one is in the causal cone of the other or not. In fact, nothing special happens for points on the boundary of the light-cone itself.

Moving to a relativistic theory, but still sticking to the one-particle, or first quantized, interpretation, we would change (201) to

$$G_0(\vec{x}, \vec{x}'; t) = \int d^3p \langle \vec{x}' | e^{-i\sqrt{\vec{p}^2+m^2}t} | \vec{p} \rangle \langle \vec{p} | \vec{x} \rangle. \tag{202}$$

A direct calculation shows that

$$G_0(\vec{x}, \vec{x}'; t) \sim e^{-m\sqrt{|\vec{x}'-\vec{x}|^2 - t^2}}. \tag{203}$$

This expresion, although relativistic does not really solve the causality issue present in (201); the propagation amplitude is small but non zero for $|\vec{x}' - \vec{x}|^2 \gg t^2$ (well outside the light cone).

Quantum field theory solves this problems by introducing antiparticles. We shall see that the culprit are the negative energy modes and that causality-violating propagation of negative energy modes is turned into causality compliant propagation thanks to antiparticles. When we ask whether an observation made at space-time point x can influence another at y, with $(x-y)^2 \leq 0$, we will find that the amplitudes due to particle and antiparticle propagation exactly cancel.

To fulfill Green's equation (see below) the factor $\theta(x^0 - y^0)$ is included in quantum mechanics in the propagator.

5.2 Propagators in field theory

We shall present the discussion in terms of a scalar field and in the next section we shall discuss the modifications for fermions and vector fields. Let us introduce the following two-point function in field theory

$$D(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{204}$$

Using elementary QM techniques, this function can be written as

$$\langle 0|\phi(\vec{x},0)e^{-i\mathcal{P}^0(x^0-y^0)}\phi(\vec{y},0)|0\rangle, \tag{205}$$

where we have defined the vacuum energy to be zero; i.e. we work with a normal ordered hamiltonian $\mathcal{H} = \mathcal{P}^0$. Taking into account that we go from QM to field theory by the replacement $\vec{x} \to \phi(\vec{x})$, the analogy with (201) is manifest.

A simple calculation shows that $D(x,y)$ is a c-number and, in fact, a function of $x-y$.

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \int \frac{d^3k}{(2\pi)^3 2E_k} e^{-ipx+iky} \langle 0|a_p a_k^\dagger|0\rangle \tag{206}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)}. \tag{207}$$

Take for instance a space-like separation, $x^0 = y^0$, $\vec{r} = \vec{x} - \vec{y}$, then

$$\int \frac{d^3p}{(2\pi)^3 2E_p} e^{-i\vec{p}\vec{r}} \sim e^{-mr}. \tag{208}$$

Again, the propagation seems to be non causal.

Taking into account that $\phi(x)$ is real field, it is actually an hermitian operation acting in the Hilbert space of the system. Thus $\phi(x)$ is an observable. Now we can pose the causality issue in very clear terms. If

$$[\phi(x), \phi(y)] = 0 \tag{209}$$

a measurement made at x cannot affect another made at y. Actually we know that this is is always the case if $x^0 = y^0$, as this is actually the content of one of the canonical commutation relations. Physically it means that we can set our initial data for a constant time slice. Let us now compute the above commutator for unequal times

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3 2E_p} \int \frac{d^3k}{(2\pi)^3 2E_k} [a_p e^{-ipx} + a_p^\dagger e^{ipx}, a_k e^{-ikx} + a_k^\dagger e^{ikx}] \tag{210}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} (e^{-ip(x-y)} - e^{-ip(y-x)}) = D(x-y) - D(y-x). \tag{211}$$

We see that indeed if $x^0 = y^0$, and changing $\vec{p} \to -\vec{p}$ in the second integral, the commutator is zero, as expected.

It is more interesting to note that $D(x-y)$ is a Lorentz invariant. This is not totally obvious, but it is so by noticing that

$$\int \frac{d^3p}{(2\pi)^3 2E_p} = \int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0) \tag{212}$$

which is manifestly invariant under proper gauge tranformations (those connected to the identity), and that e^{ipx} is by itself invariant. Now, if $(x-y)^2 < 0$ (y is outside the lightcone of x, or viceversa) we can find a Lorentz transformation such that

$$x - y \to -(x-y). \tag{213}$$

This is obviously impossible if $x - y$ is time-like, since it would reverse the time ordering of physical events. Therefore we conclude that

$$[\phi(x), \phi(y)] = 0 \qquad \text{if } (x-y)^2 < 0. \tag{214}$$

Causality in the sense above indicated is now manifest. Notice that the cancellation takes place between positive and negative energy solutions. Thus antiparticles are basic to restore causality.

Since $[\phi(x), \phi(y)]$ is a c-number, we can write

$$[\phi(x), \phi(y)] = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \tag{215}$$

Let us assume that $x^0 > y^0$,

$$\begin{aligned}
\theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle &= \theta(x^0 - y^0)\int \frac{d^3p}{(2\pi)^3 2E_p}(e^{-ip(x-y)} - e^{ip(x-y)}) \\
&= \theta(x^0 - y^0)\int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_p}e^{-ip(x-y)}|_{p^0=E_p} + \frac{1}{-2E_p}e^{-ip(x-y)}|_{p^0=-E_p}\right) \\
&= \theta(x^0 - y^0)\int \frac{d^3p}{(2\pi)^3}\int_C \frac{dp^0}{2\pi i}\frac{-1}{p^2 - m^2}e^{-ip(x-y)},
\end{aligned}$$

where C is a circuit containing the two poles at $p^0 = \pm E_p$ closed through the lowest half of the complex plane ($p^0 = -i\infty$) since $x^0 - y^0 > 0$. The result is thus

$$D_R(x-y) \equiv \theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle = \int_C \frac{d^4}{(2\pi)^4}\frac{i}{p^2 - m^2}e^{-ip(x-y)}. \tag{216}$$

$$D_R(x-y) = \theta(x^0 - y^0)(D(x-y) - D(y-x)). \tag{217}$$

This is called the "retarded" propagator. Another way of enforcing the integration circuit is by adding a small and negative imaginary part, declaring that the poles lie at $p^0 = \pm E_p - i\epsilon$ and integrating p^0 over the real line. Likewise we can consider an "advanced" propagator with poles at $p^0 = \pm E_p + i\epsilon$ which is non-zero for $y^0 > x^0$, but is of no particular relevance to us. To be precise, it is usually defined as

$$\theta(y^0 - x^0)\langle 0|\{\phi(x), \phi(y)\}|0\rangle. \tag{218}$$

We can check that the retarded propagator verifies the following differential equation

$$(\Box + m^2)D_R(x-y) = -i\delta^{(4)}(x-y). \tag{219}$$

This is proposed as an exercise.

In fact all propagators are related to solutions of this partial diferential equations (Green's equation) and they are therefore called Green's functions. Denoting by $\bar{D}(x-y)$ a generic propagator (retarded, Feynman...)

$$(\Box + m^2)\bar{D}(x-y) = -i\delta^4(x-y), \tag{220}$$

and thus all of them in momentum space obey

$$(-p^2 + m^2)\bar{D}(p) = -i, \tag{221}$$

and

$$\bar{D}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2}e^{-ip(x-y)} \tag{222}$$

The difference between the different Green functions ($D_R(x-y)$, $D_R(x-y)$, ...) lies in the prescription to handle the poles appearing in the p^0 integral. The retarded prescription, as we have seen, includes both of them in the contour.

Feyman prescription amounts to considering

$$\frac{i}{p^2 - m^2 + i\epsilon}. \tag{223}$$

Thus, poles are in $p^0 = E_p - i\epsilon$ and $p^0 = -E_p + i\epsilon$. Then

$$D_F = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \tag{224}$$

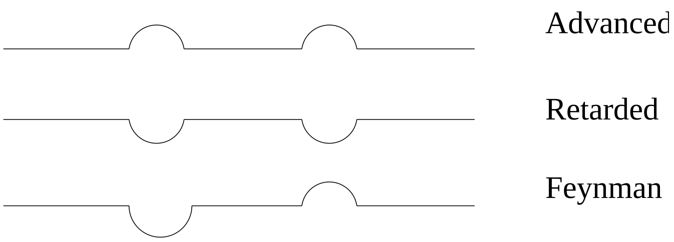
Now, if $x^0 > y^0$ we have to close the circuit in the lower half-plane, and we pick only the contribution from the pole at $p^0 = +E_p$, that is $D(x-y)$. The opposite happens if $y^0 > x^0$; in this case the non-zero contribution comes from the negative energy solution. In short

$$D_F(x-y) = \theta(x^0 - y^0) D(x-y) + \theta(y^0 - x^0) D(y-x). \tag{225}$$

This is denoted as

$$\langle 0 | T\phi(x)\phi(y) | 0 \rangle. \tag{226}$$

The different prescriptions for the poles are summarized in the following figure



The so-called Dyson prescription consists in taking

$$\frac{i}{p^2 - m^2 - i\epsilon} \tag{227}$$

with ϵ positive.

5.3 The fermion and gauge-boson propagators

The contribution from particles and antiparticles it is best discussed in a case where the distinction matters. For this reason, and also in order to discuss the peculiarities of spin 1/2 fields.

The basic definitions are the same. Green's functions are solutions of the partial differential equation

$$(i \not{\partial}_x - m)\bar{S}(x-y) = i\delta^{(4)}(x-y). \tag{228}$$

We define

$$S_+(x-y) = \langle 0|\psi(x)\bar{\psi}(y)|0\rangle, \tag{229}$$

$$S_-(y-x) = \langle 0|\bar{\psi}(y)\psi(x)|0\rangle, \tag{230}$$

and

$$S_F(x-y) = \theta(x^0-y^0)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle - \theta(y^0-x^0)\langle 0|\psi(y)\bar{\psi}(x)|0\rangle, \tag{231}$$

where the minus sign appears on account of the anticommuting character of the variables.

We can easily solve Green's equation in momentum space, the result is

$$\frac{i}{\not{k} - m}. \tag{232}$$

What this actually means is

$$\frac{i}{\not{k} - m} = i\frac{\not{k} + m}{k^2 - m^2}. \tag{233}$$

Feynman propagator is obtained by considering

$$i\frac{\not{k} + m}{k^2 - m^2 + i\epsilon}. \tag{234}$$

An alternative way of deriving the previous expressions is by directly inserting the mode expansion, for instance in

$$S_+(x-y) = \langle 0|\psi(x)\bar{\psi}(y)|0\rangle. \tag{235}$$

We have as a non-vanishing contribution

$$\int \frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3k}{(2\pi)^3 2E_k}\sum_{\sigma,\sigma'} e^{-ipx+iky}u(p,\sigma)\bar{u}(k,\sigma')\langle 0|c(p,\sigma)c^\dagger(k,\sigma')|0\rangle \tag{236}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p}\sum_{\sigma} u(p,\sigma)\bar{u}(k,\sigma)e^{-ip(x-y)}. \tag{237}$$

Likewise

$$S_-(x-y) = \langle 0|\bar{\psi}(y)\bar{\psi}(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_\sigma v(p,\sigma)\bar{v}(p,\sigma)e^{ip(x-y)}. \tag{238}$$

The sum over spinors gives $\not{k}+m$ ($\not{k}-m$) in the first (second) case.

We now see that $\langle 0|\psi(x)\bar{\psi}(y)|0\rangle$ propagates particle (positive energy solutions) from y to x while $\langle 0|\bar{\psi}(y)\psi(x)|0\rangle$ propagates antiparticles from x to y. Causality, is now the statement that

$$\{\psi(x), \bar{\psi}(y)\} \tag{239}$$

vanishes for space-like separations. This can be easily proven by using the same methods as for the bosonic case and it necessitates both both contributions, particles and antiparticles, both propagating forward in time, but with reversed momentum.

The Feynman propagator contains both contributions too

$$S_F(x-y) = \theta(x^0-y^0)S_+(x-y) - \theta(y^0-x^0)S_-(y-x), \tag{240}$$

but not simultaneosuly! If $x^0 > y^0$ we have particle propagation, while for $y^0 > x^0$ we have antiparticle propagation.

We finally turn to gauge fields. We shall be brief in this case as we have already discussed most of the subtle issues. Working in an arbitrary gauge, the corresponding Green's equation is

$$(\Box g^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu)D_{\nu\alpha}(x-y) = i\delta^\mu_\alpha\delta^{(4)}(x-y), \tag{241}$$

whose solution is, in momentum space

$$(-i)\frac{g^{\nu\alpha} - (1-\xi)\frac{k^\nu k^\alpha}{k^2}}{k^2+i\epsilon}, \tag{242}$$

where we have already adopted Feynman prescription.

5.4 Feynman interpretation of negative energy states and antiparticles

It should be clear by now that in spite of its brilliant success for spin 1/2 particles, Dirac theory of holes and particles cannot be applied to bosons. Yet, bosons have their own antiparticles. We are thus led to the following hypothesis due to Feynman:

"The emission (absorption) of an antiparticle of 4-momentum p^μ is physically equivalent to the absorption (emission) of a particle of 4-momentum $-p^\mu$"