STANDARD MODEL- Part I

D. Espriu*
Departament d'Estructura i Constituents de la Matèria and
CER for Astrophysics, Particle Physics and Cosmology
Universitat de Barcelona,
Diagonal 647, E-08028 Barcelona.

*E-mail: espriu@ecm.ub.es

1 Fields, currents, and symmetries

1.1 Conserved currents and charges

Assume that there is a field transformation $\delta\phi$ such that $L(\phi+\delta\phi) = L(\phi)$, i.e. $\delta L = 0$. The lagrangian L has then a symmetry. If L depends on ϕ only through the field itself and its first derivative, then

$$\delta L = \frac{\delta L}{\delta\phi}\delta\phi + \frac{\delta L}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) \tag{1}$$

Using the Euler-Lagrange equation

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu\phi)} - \frac{\delta L}{\delta\phi} = 0 \tag{2}$$

$$\delta L = \partial_\mu\left(\frac{\delta L}{\delta(\partial_\mu\phi)}\right)\delta\phi + \frac{\delta L}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \partial_\mu\left(\frac{\delta L}{\delta(\partial_\mu\phi)}\delta\phi\right) = 0 \tag{3}$$

This implies that there exists a conserved current

$$J^\mu = \frac{\delta L}{\delta(\partial_\mu\phi)}\delta\phi \tag{4}$$

This is the contents of Noether's theorem.

Note.- for continuous global transformations, $\delta\phi$ usually contains a small parameter that is factored out in J^μ.

Let us now assume that we have a set of N fields ϕ_i. In particle physics we shall be mostly concerned with transformations of the fields that are linear and unitary. Such a symmetry takes the form

$$\delta\phi = -i\epsilon_a T^a \phi \tag{5}$$

The T^a are a set of hermitian matrices that may be traceless or not, generators of some symmetry group G. If the group of transformations is continuous,

$$[T^a, T^b] = i f^{abc} T^c. \tag{6}$$

The constants f^{abc} are the group's structure constants. The parameters ϵ_a are real.

Then $\phi \to \phi' = \phi - i\epsilon_a T^a \phi$, and the Noether current (4) is

$$J^{\mu a} = -i\frac{\delta L}{\delta(\partial_\mu\phi)} T^a \phi \tag{7}$$

Out of the Noether current J^μ we can construct a conserved charge Q.

$$Q = \int d^3x\, J^0(x,t) \tag{8}$$

$$\frac{dQ}{dt} = \frac{d}{dt}\int d^3x J^0(x,t) = -\int d^3x \frac{\partial J^i}{\partial x^i}(x,t) = 0 \tag{9}$$

For more than one generator,

$$Q^a = \int d^3x\, J^{0a}(x,t) \tag{10}$$

Exercise.- Verify that the charges Q^a realize the commutation relations of the symmetry group; that is:

$$[Q^a, Q^b] = i f^{abc} Q^c. \tag{11}$$

Use the canonical commutation relations of field theory (E.T.C.), i.e.

$$[\phi_i(x), \phi_j(y)] = [\pi_i(x), \pi_j(y)] = 0,\; x^0 = y^0 \tag{12}$$

$$[\phi_j(x), \pi_k(y)] = i\delta_{j,k}\delta^{(3)}(\vec{x} - \vec{y}),\; x^0 = y^0 \tag{13}$$

1.2 P, C and chirality

We will now discuss two important discrete symmetries (i.e. not of the type discussed before) that are best understood by the way they act on fermions. The lagrangian of free massive fermions is

$$L(\psi) = i\bar{\psi}\, \partial\!\!\!/ \psi - m\bar{\psi}\psi. \tag{14}$$

The Euler-Lagrange equation associated to this lagrangian is the well-known Dirac equation, $i\, \partial\!\!\!/ \psi - m\psi = 0$ (without any external fields). Let us now consider the following projection operators

$$P\pm = \frac{1}{2}(1 \pm \gamma_5) \tag{15}$$

and take, for the time being, $m = 0$.
Note: $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma_5 = \gamma_5^\dagger$, $\{\gamma^\mu, \gamma^u\} = 2g^{\mu\nu}$.

Exercise.- Prove that γ_5 anticommutes with all the γ^μ and that $(\gamma_5)^2 = 1$.

We define the chirally projected fermions $\psi_{L,R}$

$$\psi_L = P_-\psi,\; \psi_R = P_+\psi \tag{16}$$

so

$$\bar{\psi}_L = \bar{\psi} P_+,\ \bar{\psi}_R = \bar{\psi} P_+ \tag{17}$$

Exercise.- Prove (17). Recall that $\bar{\psi} = \psi^\dagger \gamma^0$.

The L (left) and R (right) states contain half the Dirac fermion degrees of freedom. They are in fact helicity eigenstates for $m = 0$. To see that we write the Dirac equation in the $\vec{k} = \hat{z}$ frame; then $k^1 = k^2 = 0$, $k^0 = k^3$, so Dirac implies

$$k(\gamma^0 - \gamma^3)\psi = 0 \Rightarrow \gamma^0 \psi = \gamma^3 \psi. \tag{18}$$

The angular momentum operator (a generator of the Poincaré group) acting on spin 1/2 particles is

$$J^i = \frac{1}{4}\epsilon^{ijk}\sigma^{jk},\ \sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k] \tag{19}$$

so

$$J^3 = \frac{1}{2}\sigma^{12} = \frac{i}{2}\gamma^1\gamma^2. \tag{20}$$

Then

$$J^3 \psi_L = \frac{1}{2} i\gamma^1\gamma^2 \frac{1-\gamma_5}{2}\psi = \frac{1}{2}\frac{1-\gamma_5}{2} i\gamma^0\gamma^0\gamma^1\gamma^2\psi = \frac{1}{2}\frac{1-\gamma_5}{2}\gamma^5\psi = -\frac{1}{2}\psi_L \tag{21}$$

and $J^3\psi_R = +\frac{1}{2}\psi_R$. The chirally projected fermions are, in the massless case, helicity eigenstates. Recall that helicity is defined as

$$\frac{\vec{J}\vec{p}}{|p|}. \tag{22}$$

If $m \neq 0$, chirality and helicity do not coincide. Chirality is not a conserved quantum number for massive particles.

Since helicity involves a pseudovector, its sign must change under a parity transformation. Let us see how parity can be implemented at the operator level on fermions. Let us consider the free Dirac equation (in momentum space)

$$(\not{p} - m)\psi = (p^0\gamma^0 - p^i\gamma^i - m)\psi = 0. \tag{23}$$

Under a parity transformation: $p^0 \to p^0, p^i \to -p^i$

$$(p^0\gamma^0 + p^i\gamma^i - m)\psi' = 0. \tag{24}$$

Clearly $\psi' = e^{i\varphi}\gamma^0\psi$, and $\varphi = 0, \pi$. If the Dirac equation contains a coupling to an external vector field, this changes accordingly under parity: $A^0 \to A^0$, $A^i \to -A^i$.

Exercise.- Check that γ^0 changes the chirality of the fields. Is it true that $\psi'_L = \psi_R$? Write the mass term in the lagrangian in terms of right-handed and left-handed fields. Does the mass term break parity?

Note that for *massive* particles, chirality is not a good quantum number. The mass term turns left landed into right handed. On the other hand, gauge interactions (vector or axial-vector) always preserve helicity, while scalar and pseudoscalar interactions do not.

Exercise.- Prove it !

Next we turn to charge conjugation C. Its meaning is best seen by considering the coupling of fermions to an external gauge field. According to the minimal coupling principle, the Dirac equation reads

$$i\, \partial\!\!\!/ \psi - e\, A\!\!\!/ \psi - m\psi = 0 \tag{25}$$

(We shall often use the *covariant derivative* $D_\mu = \partial_\mu + ieA_\mu$.) Obviously the equation describing the coupling to antiparticles should be

$$i\, \partial\!\!\!/ \psi_c + e\, A\!\!\!/ \psi_c - m\psi_c = 0 \tag{26}$$

Let us find the relation between ψ and ψ_c. We complex conjugate the first equation

$$-i\gamma^*_\mu \partial^\mu \psi - e\gamma^*_\mu A^\mu \psi^* - m\psi^* = 0 \tag{27}$$

Let us now multiply by a matrix C such that

$$C\gamma^*_\mu C = -\gamma_\mu \tag{28}$$

and such that $C^2 = 1, C^\dagger C = 1$. Elementary manipulations allow us to show that

$$-iC\gamma^*_\mu C\partial^\mu C\psi - eC\gamma^*_\mu CA^\mu C\psi^* - mC\psi^* = 0 \tag{29}$$

so, clearly,

$$\psi_c = C\psi^* \tag{30}$$

It can be seen that for Dirac fermions $C = i\gamma^2$.

Exercise.- Find the implementation of charge conjugation on a complex scalar field. Is it possible to couple electromagnetism via minimal coupling to a real scalar field?

1.3 Dirac and Majorana fermions

The actual expression of spinors and Dirac-logy depends on the representation one chooses for the Dirac matrices. The usual one is the so-called Dirac representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{31}$$

Another important representation is the Majorana representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \tag{32}$$

all the γ's are purely imaginary, so $(\gamma^\mu)^* = -\gamma^\mu$. Hence we can take $C = 1$. Furthermore, the spinors that satisfy the free Dirac equation in the Majorana representation can be taken to be purely real (or purely imaginary) (simply complex-conjugate $(i\not\partial - m)\psi$). Let's assume we take them to be real, then

$$\psi_c = \psi \tag{33}$$

The real solutions describe fermionic particles that are their own antiparticles. They are called Majorana fermions. They obviously contain half the degrees of freedom. Notice that the argument is no more correct if we add an external electromagnetic field to the Dirac equation — real and imaginary parts are no longer solutions separately. Clearly Majorana fermions can be described in any representation of the Dirac matrices, but the Majorana representation is the most natural.

It is easy to see that the mass term for Majorana fermions can be written in Majorana representation as

$$\chi_1^T \sigma^2 \chi_2 + \chi_2^T \sigma^2 \chi_1,\ \psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{34}$$

It breaks $U(1)$ gauge invariance as this involves a *complex* phase.

Exercise.- Verify that the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & \sigma^2 \\ \sigma^2 & -I \end{pmatrix} \tag{35}$$

moves from Dirac to Majorana representation, i.e. $\gamma_M = U\gamma_D U^\dagger$.