10 Logs in QCD

Before continuing our discussion, it is convenient to make a short theoretical digression. We have seen how the scaling of α_s with the energy is logarithmic. Logs in fact play a very crucial role in Quantum Field Theory. Let us then stop and think for a second which is the origin of these logarithmic terms.

Actually there are two types of logarithms in a Field Theory such as QCD

$$\log\frac{q^2}{\mu^2} \qquad \log\frac{q^2}{\lambda^2} \tag{201}$$

They have very different origin. μ is some renormalization or subtraction scale, while λ^2 can be some external momentum squared or a small $(\text{mass})^2$ that we have given by hand to the gluon. The first type are associated to ultraviolet divergent integrals (integrals with a bad behaviour when the internal momentum is large). The second type are infrared logs and are related to Feynman diagrams with a bad behaviour when one or more external momenta vanish. Logs of the renormalization scale appear in any renormalizable Field Theory after renormalization. On the contrary, infrared logs appear whenever a theory has massless particles in the spectrum (such as photons or gluons).

A given Feyman diagram can give rise to both type of singularities at the same time. This is illustrated in fig. 14

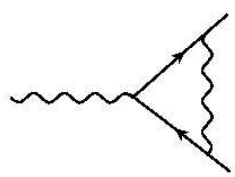

Figure 14: Integral yielding an IR and a UV log.

There actually two classes of infrared logs caused by massless particles. The so-called infrared divergences arise from the presence of a *soft massless* particle ($k^\mu \to 0$). For instance in the process $e^+e^- \to \mu^+\mu^-$ at the one loop level (fig. 14) we have to compute the integral

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2[(p_1+k)^2-m^2][(p_2+k)^2-m^2]}. \tag{202}$$

When $p_1^2 = p_2^2 = m^2$ the integral behaves for $k^\mu \to 0$ as

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^4}. \tag{203}$$

and diverges. This divergence is unphysical so it must be cancelled by something else.
The Bloch-Nordsieck theorem states that in inclusive enough cross-sections the infrared logs cancel. What do we mean by 'inclusive enough'? A detector will not be able to discern a 'true' muon from a muon accompanied by a soft enough photon (with $\vec{k} \to 0$). Therefore, in addition of the diagram shown in fig. 14 we have to consider diagrams where a soft photon is radiated by the muon, square the modulus of the amplitude and integrate over the available phase space (which actually depends on the experimental cut). When this is done the result is infrared finite. The relevant diagrams are depicted in fig. 15

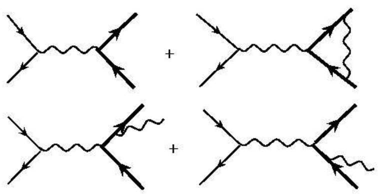

Figure 15: Real and virtual photons have to be included for IR safe results.

The other type of infrared logs are called mass singularities. They occur in theories with massless particles because two *parallel massless* particles have an invariant mass equal to zero

$$k^2 = (k_1 + k_2)^2 = \|(\omega_1 + \omega_2, 0, 0, \omega_1 + \omega_2)\| = 0. \tag{204}$$

The appeareance of such a mass singularity is illustrated in fig. 16

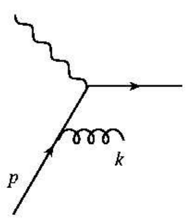

Figure 16: Diagram with a mass singularity.

$$\frac{1}{(p-k)^2} = \frac{1}{p^2 + k^2 - 2k^0p^0 + 2k^0p^0 \cos\theta}, \tag{205}$$

the denominator vanishes when we set all particles on shell ($p^2 = k^2 = 0$) and $\theta \to 0$ (i.e. $\vec{k}$ is parallel to $\vec{p}$). Even if one of the two particles is massive there is a singularity, provided the 3-momenta are parallel.

The Kinoshita-Lee-Nauenberg theorem ensures that for inclusive enough cros section the mass singularities also cancel. Both for mass singularities and for infrared divergences there is a trade-off between λ^2, the infrared regulator of a massless particle, and the energy and angle resolution of the inclusive cross section ΔE, $\Delta\theta$.
In practice, it is better to regulate the infrared logs using dimensional regularization (introducing λ^2 leads to difficulties with gauge invariance). Real gluon emission diagrams are regulated by performing the phase space integration in n dimensions.

There is in fact a lot of physical insight hidden in the infrared logs. We have seen that the contribution from the diagram in fig. 14 is infrared divergent, i.e. *infinite*. Yet, physical arguments tell us that the probability of finding a 'bare' isolated muon should be *zero*. We know this because detectors are unable to tell apart a muon from a muon plus one soft photon or indeed from a muon plus any number of soft photons. Infrared divergences in QED can be summed up and then one sees that the probability of finding an isolated muon is indeed zero and not infinite as the one loop diagram led us to believe. Whenever a Feynman diagram is infrared divergent it means that we have forgotten something relevant.

In QED the situation is under theoretical control. If M is some amplitude, in perturbation theory we expand the corresponding matrix element M in perturbation theory

$$M = \sum_{n=0}^{\infty} M_n. \tag{206}$$

A calculation shows that the amplitude has the following structure

$$\begin{aligned} M_0 &= m_0, \\ M_1 &= m_0 \alpha B + m_1, \\ M_2 &= m_0 \frac{(\alpha B)^2}{2} + m_1 \alpha B + m_2, \\ &\dots \end{aligned} \tag{207}$$

The quantities m_n are IR-finite, while B is IR-divergent. The previous series can be summed up

$$M = \exp(\alpha B) \sum_{n=0}^{\infty} m_n, \qquad m_n \sim \alpha^n, \tag{208}$$

and B can be obtained just from the lowest order diagram. Introducing an IR cut-off λ, $B \sim -\log m^2/\lambda^2$, which indeed shows that when we remove the cut-off the probability of finding an isolated charged fermion is zero in QED.

The addition of soft photons changes that result dramatically by multiplying the total amplitude by a factor $\sim (\Delta E/\lambda)^2$. There is a trade between the infrared regulator and $\Delta E, \Delta\theta$. The latter are, of course, detector- dependent quantities. The final result is proportional to $(\Delta E/m)^2$, both of them physical quantities. This shows that when $\Delta E \to 0$ the amplitude is strictly zero, in accordance with physical expectations.

Although only partial results exist, it is believed that a similar exponentiation takes place in QCD. Due to the confinement subtleties it is unclear whether the suppression factor is compensated by radiation of soft gluons. Even if this compensation does actually take place that would not disprove confinement, only that confinement would have nothing to do with the structure of infrared singularities of the theory.

10.1 Jets and α_s

The previous discussion can be summarized in the following way: due to IR singularities one is forced to consider cross sections not of individual particles in the final state, but rather of bunches of particles, each 'hard' quark and gluon surrounded by a 'soft' cloud of gluons and, perhaps, quarks. We will call these bunches 'jets'.

The Bloch-Nordsieck and Kinoshita-Lee-Nauenberg theorems guarantee the finiteness of the cross-sections. We have to define an energy and angle resolution. For instance, if p is the momentum of a primary quark we can impose that the energy of each soft particle in its jet satisfies $k_i^0 < \epsilon p_0$ and also that $\arg(\vec{p}, \vec{k}_i) < \delta$. We will get singularities of the form $\alpha_s \log\epsilon \log\delta$ when $\epsilon, \delta \to 0$. The specific details depend on the precise definition of the jet.
The situation is unfortunately even more involved because hadrons and not quarks and gluons are detected. The evolution of the quarks and gluons produced at high momentum transfer is perturbative at first, until the average separation of the particles becomes $\mathcal{O}(\Lambda_{QCD}^{-1})$. Then the confining potential (and the string picture) takes over. Eventually one is forced for anything other than fully inclusive observables (such as R_{had}) to introduce fragmentation and hadronization models to compute the observable cross-sections. Any observable that is not fully inclusive is described by a convolution of two very different types of physics

$$\text{Observable} = \text{Perturbative} \otimes \text{Non} - \text{perturbative} \tag{209}$$

The perturbative part is, in principle, calculable in QCD as a power series in α_s. It is affected by some 'theoretical' uncertainties since most observables have been calculated up to next to leading order, and not beyond, and higher order effects can be important. In addition there is the issue of the choice of an adecuate renormalization scale, which sometimes is far from obvious.

The non-perturbative part has to be modelled. Its relation to QCD and its parametres (such as α_s) is unclear. A considerable amount of cross-checking and experimental feed-back is required. In general, the more inclusive the observable is the smaller the unknowns coming in from the non-perturbative part are. A number of observables are widely used where the dependence on the hadronization model is believed to be under control such as thrust, or energy-energy correlations.

Thrust is defined as

$$T = \max \frac{|\sum_i \vec{p}_i \vec{n}_T|}{\sum_i |\vec{p}_i|} \tag{210}$$

$\vec{n}_T$ is the thrust axis, which is varied to maximize T. T takes values between 0.5 (spherical) to $T = 1$ (complete alignment). To quote a figure, the average LEP value is close to 0.94, higher than in any previous experiment; events are well aligned with the momentum of the primary quark. As a consequence, it is much easier to count jets at LEP than in any previous machine.