11 Deep inelastic scattering

11.1 Free parton model

The counterpart of having an effective coupling constant which grows at low energies is that the theory becomes simple at high energies, making perturbative calculations possible (at least *some* of them). Indeed a brilliant confirmation of the existence of nearly free constituents inside the nucleon was provided more than twenty years ago by a series of experiments carried out at SLAC. Then it became possible to scatter electrons off nucleons in fixed target experiments with a typical momentum transfer $\sim 1 - 10$ (GeV)^2, a kinematical range unexplored until that time. The kinematics of Deep Inelastic Scattering (DIS) processes is shown in fig. 17.

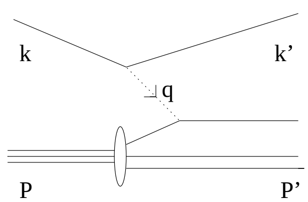

Figure 17: The kinematics of deep inelastic scattering.

The virtual intermediate boson is far off its mass-shell and scatters off a quark or gluon in a time of $\mathcal{O}(1/\sqrt{-q^2})$. Typically quarks and gluons are themselves off-shell by an amount of $\mathcal{O}(\Lambda_{QCD})$. After the scattering the outgoing particles recombine into hadrons in a time of $\mathcal{O}(1/\Lambda_{QCD})$. Thus Deep Inelastic is a two-step process

- Short distance scattering occurs with a large momentum transfer. Well described by perturbation theory.
- Outgoing particles recombine. Not calculable in perturbation theory.

However, the second part can be side-stepped all together for fully inclusive rates. Then perturbation theory is adecuate to describe many features of DIS.

If we place ourselves in the center of mass of the hadron and virtual intermediate boson both particles move very fast towards each other. Whatever components the hadron contains they will all have moments parallel to P^μ, up to transversal motion of $\mathcal{O}(\Lambda_{QCD})$. Let us write

$$p^\mu = xP^\mu. \tag{211}$$

The squared CM energy of the lepton and proton constituent will be

$$\hat{s} = (xP + k)^2 \simeq 2xPk \simeq xs. \tag{212}$$

We neglect masses (as well as the fact that constituents are off-shell by $\mathcal{O}(\Lambda_{QCD})$) The final momentum of the constituent is $xP + q$. Therefore

$$0 \simeq (xP + q)^2 \simeq 2xPq + q^2, \tag{213}$$

so $x = -q^2/2Pq$. If ν is the energy transfer in the LAB system, we can also write

$$x = \frac{-q^2}{2\nu m_N} \tag{214}$$

m_N being the nucleon mass. It is convenient to introduce

$$y = \frac{Pq}{Pk} = 1 - \frac{Pk'}{Pk} \tag{215}$$

In the lab frame $y = \nu/E$ and $0 \leq y \leq 1$. y is thus the relative energy loss of the colliding lepton.

Let us for the time being ignore altogether QCD interactions and let us assume that constituents of the nucleons (which we will call partons) are free. DIS will then be described by an incoherent sum over elementary processes. The partonic differential cross sections in the LAB frame will be

- νq, $\overline{\nu} q$-scattering

$$\frac{d\hat{\sigma}_\nu}{dy} = (\frac{g^2}{4\pi})^2 \frac{\pi m E}{(q^2 - M_W^2)^2}[g_L^2 + g_R^2(1-y)^2], \tag{216}$$

$$\frac{d\hat{\sigma}_{\overline{\nu}}}{dy} = (\frac{g^2}{4\pi})^2 \frac{\pi m E}{(q^2 - M_W^2)^2}[g_R^2 + g_L^2(1-y)^2]. \tag{217}$$

- eq-scattering

$$\frac{d\hat{\sigma}_e}{dy} = Q^2 \frac{4\pi\alpha^2 m E}{q^4}[1 + (1-y)^2]. \tag{218}$$

The neutral current sector is dominated by γ exchange below $q^2 = M_Z^2$, so we have not bothered to include Z exchange. In (218) Q is the quark electric charge (in units of e) and m is the target mass. Since $p^\mu = xP^\mu$, we just take $m = xm_N$. Then, for instance,

$$\frac{d^2\hat{\sigma}_e}{dxdy} = Q^2 \frac{4\pi\alpha^2 x m_N E}{q^4}[1 + (1-y)^2]. \tag{219}$$

Let $u(x)dx, d(x)dx, ...$ be the number of $u, d, ...$ quarks with momentum fraction between x and $x+dx$ in a nucleon. Then $xu(x), xd(x), ...$ will be the fraction of the nucleon momentum carried by $u, d, ...$ quarks. We, of course, identify quarks with partons and, since we assume that they are free, proceed to sum incoherently over the different scattering possibilities. For instance in $ep \to eX$

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi\alpha^2}{s}\frac{1+(1-y)^2}{xy^2}[\frac{4}{9}(u(x)+\overline{u}(x)) + \frac{1}{9}(d(x)+\overline{d}(x)) + \frac{1}{9}(s(x)+\overline{s}(x))]. \tag{220}$$

(We neglect here the possible contribution from the sea of heavy quarks in the nucleon.) Other DIS processes weigh differently quarks and antiquarks. For instance, in $\nu p \to \mu X$ if $-q^2 \ll M_W^2$ we have

$$\frac{d^2\sigma}{dxdy} = x\frac{G_F^2 s}{\pi}[c_c^2 d(x) + s_c^2 s(x) + \overline{u}(x)(1-y)^2], \tag{221}$$

with $c_c = \cos\theta_c$, $s_c = \sin\theta_c$, the cosinus and sinus of the Cabibbo angle, respectively.

The parton distribution functions (PDF) $q(x)$ are quantities which are not calculable within perturbative QCD, as we will see.

Probably the first thing that one learns is that gluons are very important. From the SLAC-MIT data

$$Q = U + D + S = \int_0^1 dxx(u(x) + d(x) + s(x)) \simeq 0.44, \tag{222}$$

$$\bar{Q} = \bar{U} + \bar{D} + \bar{S} = \int_0^1 dxx(\bar{u}(x) + \bar{d}(x) + \bar{s}(x)) \simeq 0.07. \tag{223}$$

The total fraction of momentum carried by quarks (and antiquarks) is only about 50% The rest is carried by gluons (parametrized by a PDF $g(x)$), showing that although the naive quark model works very well is just a gross simplification as a model of hadrons, at least at large $-q^2$. In fact we know the asymptotic values of (222) and (223) based in the equipartition of energy in a free theory (since, asymptotically, QCD is free)

$$\int_0^1 dxxq(x) \to \frac{3N_f}{16+3N_f} \qquad \int_0^1 dxxg(x) \to \frac{16}{16+3N_f}. \tag{224}$$

From the above limiting values we see and at higher energies the total momentum carried by *constituent* or *valence* quarks diminishes and that an equally important role is played by particles from the Dirac sea of the nucleon.

Exercise.- Could you justify on some physical arguments these limiting values?

Another example where the quark model fails to describe some basic features of hadrons is provided by the 'spin of the proton' problem. μ-scattering on polarized targets shows that the fraction of the total spin of the proton that can naively be associated to constituent quarks is surprisingly small. We shall not dwelve on this matter further here.

Nevertheless, there are some obvious sum rules for the parton distribution functions which can ultimately be explained in terms of the quark model. For the proton

$$\int_0^1 dx(u(x) - \bar{u}(x)) = 2, \tag{225}$$

$$\int_0^1 dx(d(x) - \bar{d}(x)) = 1, \tag{226}$$

$$\int_0^1 dx(s(x) - \bar{s}(x)) = 0. \tag{227}$$

On QCD grounds we expect that this free parton model description of the hadrons becomes more and more accurate when $-q^2 \to \infty$, $\nu \to \infty$, while keeping x fixed. This limit is known as Bjorken scaling and in the strict $-q^2 = \infty$ limit everything depends just on x.

11.2 DIS in QCD

Let us now try to rederive the previous results in a more theoretical setting. Let us consider for instance νp scattering. Then

$$\frac{d^2\sigma}{d(-q^2)d\nu} = \frac{G_F^2 m_N}{\pi s^2} L^{\mu\nu} H_{\mu\nu}, \tag{228}$$

where

$$L^{\mu\nu} = \frac{1}{8}\mathrm{Tr}[\gamma^\mu(1-\gamma_5)\gamma^\alpha\gamma^\nu(1-\gamma_5)\gamma^\beta]k^1_\alpha k^2_\beta \tag{229}$$

is the trace over the leptonic external lines, and $H_{\mu\nu}$ is given by

$$\sum_X \langle P|J_\mu(0)|X(P')\rangle\langle X(P')|J_\nu(0)|P\rangle = \int d^4z e^{iqz}\langle P|J_\mu(z)J_\nu(0)|P\rangle, \tag{230}$$

which is just $\mathrm{Im}\Pi_{\mu\nu}(q)$, with

$$\Pi_{\mu\nu}(q) = \int d^4z e^{iqz}\langle P|TJ_\mu(z)J_\nu(0)|P\rangle. \tag{231}$$

We decompose $H_{\mu\nu}$ as

$$H_{\mu\nu} = -g_{\mu\nu}F_1 + \frac{P_\mu P_\nu}{\nu m_N}F_2 + \frac{i}{2\nu m_N}\epsilon_{\mu\nu\rho\sigma}P^\rho q^\sigma F_3 \tag{232}$$

(If we assume that we are working with non-polarized targets P and q are the only vectors at our disposal.) F_1, F_2 and F_3 are called the nucleon structure functions. Using the kinematical relations $x = -q^2/2\nu m_N$ and $y = 2m_N\nu/s$ we get

$$d(-q^2)d\nu = \nu s dx dy, \tag{233}$$

$$\frac{d^2\sigma}{dxdy} = \frac{G_F^2 s}{2\pi}[F_1 xy^2 + F_2(1-y) - F_3 xy(1-\frac{y}{2})]. \tag{234}$$

Let us now compare with the free parton model. We see that (restoring the νp index, to make apparent that the structure functions are process dependent)

$$F_1^{\nu p}(x) = c_c^2(\bar{u}(x) + d(x)) + s_c^2(s(x) + \bar{u}(x)), \tag{235}$$

$$F_2^{\nu p}(x) = 2xc_c^2(\bar{u}(x) + d(x)) + 2xs_c^2(s(x) + \bar{u}(x)), \tag{236}$$

$$F_3^{\nu p}(x) = 2c_c^2(\bar{u}(x) - d(x)) + 2s_c^2(-s(x) + \bar{u}(x)). \tag{237}$$

For other processes the actual expressions may vary but the structure functions are always linear combinations of the parton distribution functions, i.e. $F_2(x) = x\sum_f \delta_f q_f(x)$, etc.

Note that in the free parton model

$$F_L(x) = F_2(x) - \frac{F_1(x)}{2x} = 0. \tag{238}$$

This is the Callan-Gross relation, which actually is not an exact one; it gets modified when the q^2 dependence is included, i.e. we depart from the strict $-q^2 = \infty$ limit.

An exact sum rule, which is easily expressed in terms of the structure function $F_2(x)$ was given by Adler

$$\int_0^1 \frac{dx}{x}(F_2^{\bar{\nu}I} - F_2^{\nu I}) = 4\langle I_3\rangle \tag{239}$$

where I_3 is the third component of the target of isospin I. Other sum rules are not exact, except in the strict free parton model, but their violations are computable within perturbative QCD. Two instances are the Gross-Llewellyn-Smith sum rule

$$\frac{1}{2}\int_0^1 dx(F_3^{\nu p} + F_3^{\bar{\nu}p}) = \int_0^1 dx(u(x) - \bar{u}(x) + d(x) - \bar{d}(x)) + \mathcal{O}(\alpha_s) = 3 + \mathcal{O}(\alpha_s), \tag{240}$$

and the Gottfried sum rule

$$\int_0^1 \frac{dx}{x}(F_2^{\mu p} - F_2^{\mu n}) = \frac{1}{3} + \mathcal{O}(\alpha_s) + \int_0^1 dx(\bar{u}(x) - \bar{d}(x)). \tag{241}$$

Violations to the different sum rules are a theoretically clean way to extract α_s. Of course in practice things are difficult because the sum rules involve an integral over all values of x, which is always poorly known in some range of the integrand and extrapolations are needed.

If we assume that α_s is extracted from some other source, the Gottfried sum rule provides some interesting information on the sea contents of light antiquarks in the nucleon. The collaboration NMC has determined that for the proton

$$\bar{U} - \bar{D} \equiv \int_0^1 dx(\bar{u}(x) - \bar{d}(x)) = -0.15 \pm 0.036. \tag{242}$$

The isospin violation is larger at low values of x ($x \leq 0.2$) and also at low values of Q^2, which hints that long-distance physics (quite remote from perturbative QCD) is called for. The enhancement of d-type antiquarks is confirmed by other experiments. For instance Na51 finds at $x = 0.18$ $\bar{u}/\bar{d} = 0.51 \pm 0.09$, while NuSeaC finds at $Q^2 = 7.4$ GeV^2 that $\bar{U} - \bar{D} = -0.10 \pm 0.024$. Could this be due to the exclusion principle which makes harder for u-type antiquarks to appear in the sea? It could be, but it is very difficult to come up with quantitative results. A partial understanding is provided by the chiral quark model, including pion exchange, but this type of physics is still very poorly understood.

11.3 Scaling violations

It is plain clear from the data that there is some $Q^2 = -q^2$ dependence in the structure functions. In other words, there are violations of Bjorken scaling and actually $F_f = F_f(x, Q^2)$. The free parton model is not completely correct (no big surprise, of course). Our job is to try to understand these violations in the framework of QCD.

Let us assume that we have isolated a parton with initial momentum $p = xP$. The probability of finding such a parton is given by $q(x)$. At the parton level the structure function F_2 is just $\hat{F}_2 = x\delta_f$ (δ_f is the appropriate charge. At the proton level, however, this partonic cross-section has to be multiplied by the probability of finding the parton with momentum fraction x, i.e. by $q(x)$. We write this in the form

$$F_2(x) = x\delta_f \int_0^1 d\xi q(\xi)\delta(x - \xi) = x\delta_f \int_0^1 \frac{d\xi}{\xi} q(\xi)\delta(1 - \frac{x}{\xi}). \tag{243}$$

At $\mathcal{O}(\alpha_s)$ many diagrams contribute. They are given in figure 19. We are looking for scaling violations and, therefore, we must look for logs. In other words we must investigate ultraviolet, infrared and mass singularities of any kind. It turns out that ultraviolet singularities

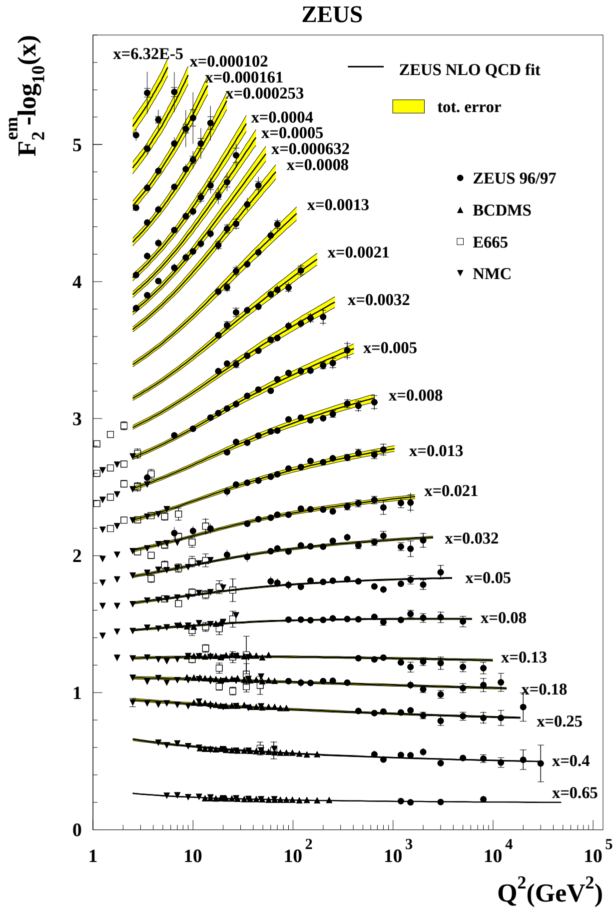

Figure 18: Evidence for scaling violations.

are proportional to the free result and simply renormalize δ_f. Infrared divergences cancel amongst all diagrams. Only the mass singularity present in diagram (d) when the momentum of the gluon is parallel to that of the gluon survives. Keeping only logarithmic terms, the calculation at $\mathcal{O}(\alpha_s)$ amounts to the replacement

$$\delta(1-\frac{x}{\xi}) \to \delta(1-\frac{x}{\xi}) + \frac{\alpha_s}{2\pi}P(\frac{x}{\xi})\log\frac{Q^2}{\lambda^2}, \tag{244}$$

where λ^2 is an infrared regulator and

$$P(z) = C_F[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)]. \tag{245}$$

The $1/(1-z)_+$ distribution is defined as

$$\int_0^1 \frac{1}{(1-z)_+}f(z)dz \equiv \int_0^1 \frac{f(z)-f(1)}{1-z}dz \tag{246}$$

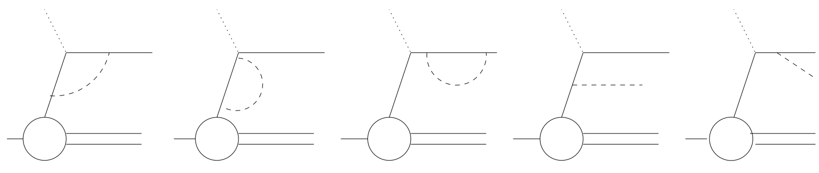

Figure 19: Contribution at $\mathcal{O}(\alpha_s)$ to the relevant DIS subprocess.

Only the logarithmic term is retained for this discussion. Now we understand the reason for writing things in apparently such a complicated way. First of all, scaling violations appear through the contribution of real soft particles. ξ is the original momentum fraction of the nucleon carried by the parton, which is reduced to $x \leq \xi$ after the emission of the soft gluon. The hard scattering takes place with the parton carrying fraction x. All along the discussion, the transverse motion of the partons inside the target is neglected as well as are all masses. Then

$$F_2(x) = x\delta_f[q_0(x) + \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \frac{\alpha_s}{2\pi} P(\frac{x}{\xi}) \log \frac{Q^2}{\lambda^2}]. \tag{247}$$

In addition we have replaced $q(x)$ by $q_0(x)$, the bare PDF. If we now define the renormalized PDF by

$$q(x, \mu^2) = q_0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} q_0(x) P(\frac{x}{\xi}) \log \frac{\mu^2}{\lambda^2}, \tag{248}$$

we can write

$$F_2(x, Q^2) = x\delta_f[q(x, \mu^2) + \int_x^1 \frac{d\xi}{\xi} q(x, \mu^2) \frac{\alpha_s}{2\pi} P(\frac{x}{\xi}) \log \frac{Q^2}{\mu^2}]. \tag{249}$$

No doubt the similarity with the usual renormalization process did not go unnoticed. Now the infrared regulator has been eliminated (hidden in the bare PDF, note that this is intrinsically an IR ill-defined object because of the soft gluon emission) at the expense of introducing a renormalization-scale dependence. These are the sought after scaling violations.

11.4 Altarelli-Parisi Equations and Λ_{QCD}

At this point it is convenient to introduce the variable $t = \frac{1}{2} \log \mu^2/\Lambda_{QCD}^2$. It then follows from (248) that

$$\frac{\partial}{\partial t} q(x, t) = \frac{\alpha_s(t)}{\pi} \int_x^1 \frac{d\xi}{\xi} q(\xi, t) P(\frac{x}{\xi}), \tag{250}$$

which immediately translate into differential equations for the structure functions themselves. These are the Altarelli-Parisi equations. They summarize the rate of change of the parton distribution functions with t.

We define the moments of the PDF's by

$$q(n,t) = \int_0^1 dx x^{n-1} q(x,t). \tag{251}$$

Introducing the anomalous dimension γ_n as

$$\gamma_n = \int_0^1 dx x^{n-1} P(x), \tag{252}$$

the convolution over the fractional momentum ξ transforms into a product

$$\frac{\partial}{\partial t} q(n,t) = \frac{\alpha_s(t)}{\pi} \gamma_n q(n,t). \tag{253}$$

This leads to following scaling behaviour for the moments of the structure functions

$$F_2(n,Q^2) = F_2(n,Q_0^2) \left(\frac{\alpha_s(Q_0)}{\alpha_s(Q)} \right)^{\frac{\gamma_n}{\beta_1}}, \tag{254}$$

which is our final expression. Experiments agree on the whole very nicely with the scaling violations predicted by QCD. Taking into account all the subtle points of Quantum Field Theory that have gone into the analysis, this provides a beautiful check of the theoretical framework.

Exercise.- Prove eq. (254).

We have been considering F_2, but the same procedure can be repeated for any structure function. The expression (254) amounts to resumming the leading logs obtained by iteration of soft collinear gluons. The diagram is the one shown in figure 20.

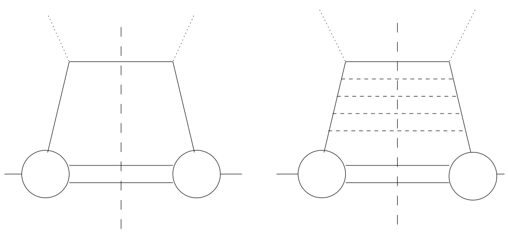

Figure 20: Handbag and ladder diagrams.

As a simplifying hypothesis we have neglected mixing. In fact, the evolution equation is a $(2N_f + 1) \times (2N_f + 1)$ matrix, involving quarks and gluons. In the flavour singlet case life is more complicated; there is mixing with gluon operators and therefore one must also consider gluon parton distribution functions as well

$$\frac{\partial q(x,t)}{\partial t} = \frac{\alpha_s(t)}{\pi} \int_x^1 \frac{dy}{y} [q(y,t) P_{qq}(\frac{x}{y}) + g(y,t) P_{gq}(\frac{x}{y})] \tag{255}$$

$$\frac{\partial g(x,t)}{\partial t} = \frac{\alpha_s(t)}{\pi}\int_x^1 \frac{dy}{y}[g(y,t)P_{gg}(\frac{x}{y}) + q(y,t)P_{qg}(\frac{x}{y})] \tag{256}$$

The detailed form of the Altarelli-Parisi kernels at leading and NLO order can be found in. No complete calculation exists yet at the NNLO to my knowledge, just some partial results. Note that P_{qg} and P_{gq} are flavour independent (in the massless limit) and that $P_{q_iq_j} \propto \delta_{ij}$ at leading order.

It is important to realize that the Altarelli-Parisi equations are not exact. They take into account the perturbative contribution only (and this up to a given order in perturbation theory). They also neglect transverse motion, which leads to corrections of $\mathcal{O}(\Lambda_{QCD}/Q^2)$ to the leading results. These are more easily dealt with in the perhaps more rigorous (but more cumbersome) treatment based in the Operator Product Expansion, which will not be discussed here. Target mass corrections should be equally taken into account. They are particularly important near thresholds (such as the charm and bottom thresholds). For instance the proper treatment of thresholds is highly relevant for HERA, since a big chunk of the data comes from a region close to these thresholds. And, of course, mass corrections are important for charm PDF from the sea, which have actually been recently measured.

The analysis of Deep Inelastic Scattering based on the Altarelli-Parisi equations (or, alternatively, on the Operator Product Expansion) has been one of the most clear tests of perturbative QCD and traditionally the best way of determining α_s, which, as we have seen, enters in the scaling violations. However, at present the value of $\alpha_s(M_Z)$ extracted from Z-physics is equally accurate if not more.

11.5 Parton distribution functions

We do not know in general how to compute the parton distribution functions, even for $-q^2 \to \infty$. Only their evolution can be reliably computed either through the Operator Product Expansion of the use of the Altarelli-Parisi equations and this for large enough values of $-q^2$. The scaling behaviour is governed by the anomalous dimensions. At leading order they are

$$\gamma_{qq}(j) = C_F[-\frac{1}{2} + \frac{1}{j(j+1)} - 2\sum_{k=2}^{j}\frac{1}{k}], \tag{257}$$

$$\gamma_{qg}(j) = T_R[\frac{2+j+j^2}{j(j+1)(j+2)}], \tag{258}$$

$$\gamma_{gq}(j) = C_F[\frac{2+j+j^2}{j(j^2-1)}], \tag{259}$$

$$\gamma_{gg}(j) = 2C_A[-\frac{1}{12} + \frac{1}{j-1} + \frac{1}{(j+1)(j+2)} - \sum_{k=2}^{j}\frac{1}{k}] - \frac{2N_f}{3}T_R, \tag{260}$$

where C_F, T_R and C_A are group-theoretical factors.

An interesting issue is the behaviour of the parton distribution functions at the endpoints $x = 0$ and $x = 1$. The large n behaviour of the moments probes the $x \to 1$ region. Since it is natural to expect that at the kinematical boundaries the parton distribution functions vanish, one can make the following ansatz for $x \to 1$

$$q(x, Q^2) \sim A(Q^2)(1-x)^{\nu(\alpha_s(Q^2))-1}. \tag{261}$$

Demanding that eq. (261) fulfills the q^2 evolution equation leads to

$$A(Q^2) = A_0\frac{[\alpha_s(Q^2)]^{-d_0}}{\Gamma(1+\nu(\alpha_s(Q^2)))} \qquad \nu(\alpha_s) = \nu_0 - \frac{16}{33-2N_f}\log\alpha_s(Q^2), \tag{262}$$

$$d_0 = \frac{16}{33-2N_f}(\frac{3}{4} - \gamma_E). \tag{263}$$

Likewise, for the gluons we have

$$g(x, Q^2) \sim A_0'\frac{[\alpha_s(Q^2)]^{-d_0}}{\Gamma(2+\nu(\alpha_s(Q^2)))}\frac{(1-x)^{\nu(\alpha_s(Q^2))}}{\log(1-x)}. \tag{264}$$

The constants A_0, A_0' and ν_0 are not calculable on perturbative QCD and depend on the specific operator. d_0 is universal.

When $x \to 1$ the gluon distribution functions approach zero more rapidly than the quark ones. For large values of x the quark contents of nucleons is the relevant one. Second order corrections to this asymptotic behaviour can be derived in a similar way and are known. It turns out that the correction is arbitraryly large if one gets sufficiently close to $x = 1$. This is because the collinear gluon is, in addition, soft in that exceptional configuration thus giving rise to a $\log(1-x)$ singularity. Multiple emission is then kinematically favoured, since the log overcomes the α_s suppression.

For small values of x the opposite behaviour takes place, the gluon distribution function eventually becomes dominant. At LHC the cross-section will be greatly dominated by low-x physics and the important process there will be gluon-gluon scattering. At the current

Tevatron run the quark contents of protons and antiprotons is still dominant. Let us see why gluons dominate completely at low x.

The key point is the appeareance of the singularity for $j = 1$. Indeed, as $j \to 1$

$$\gamma_{gg}(j) \sim \frac{2N}{j-1}. \tag{265}$$

Then

$$g(j,t) = g(j,t_0)\exp[\frac{N}{\pi\beta_1(j-1)}\log\frac{t}{t_0}]. \tag{266}$$

Then we proceed to evaluate $g(x,t)$ by performing an inverse Mellin transform

$$xg(x,t) = \frac{1}{2\pi i}\int_C djx^{1-j}g(j,t). \tag{267}$$

The integration circuit is a line in the direction of the imaginary axis in the complex j plane, to the right of the $j = 1$ singularity. The saddle point method can now be used provided that $\log(1/x)$ is large and this is the reason why this procedure gives only the small x behaviour. Working things out we see that the gluon parton distribution function for low x behaves as

$$g(x) \sim \frac{1}{x}\exp\sqrt{C(Q^2)\log\frac{1}{x}}, \tag{268}$$

where $C(Q^2)$ is calculable. Unfortunately, this answer is not totally satisfactory because something must stop the growth in $g(x)$ for low x, or else one runs into unitarity problems sooner or later, and thus eq. (268) it is not credible all the way to $x = 0$. Technically speaking, there must be corrections that destabilize the saddle point solution. Physically, the uncontrolled growth of the gluon distribution is an infrared unstability. The density of soft gluons is too large. Shadowing and non-linear evolution equations are the buzzwords here.

Exercise.- Show that if we define $\xi = \frac{N}{\pi\beta_1}\log\frac{t}{t_0}$ and $Y = \log\frac{1}{x}$, the saddle point corresponds to $j_0 = 1 + \sqrt{\frac{\xi}{Y}}$ and prove (268).

The double scaling limit (high Q^2, low x) is well supported by the data.

Except in these two limiting cases, PDF's have to be parametrized. The way one proceeds is by proposing a given parametrization at some reference value Q_0^2, then evolve to all desired values of Q^2 using the Altarelli-Parisi equations, then perform a global fit of the parameters describing the PDF and, at the same time, determine α_s.