3 Lagrangian and gauge invariance

3.1 Basic concepts

The lagrangian of a non-abelian gauge theory including n fermions in the fundamental representation of a given gauge group is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i \sum_{j=1}^{n} \bar{\psi}^\alpha_j \gamma^\mu D_{\mu\,\alpha\beta} \psi^\beta_j - \sum_{j=1}^{n} m_j \bar{\psi}^\alpha_j \psi_{j\alpha}, \tag{45}$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu, \\ D_{\mu\,\alpha\beta} &= \delta_{\alpha\beta} \partial_\mu - i g T^a_{\alpha\beta} W^a_\mu. \end{aligned} \tag{46}$$

Notice the conspicuous absence of a mass term for the gauge fields. The generators T^a belong to the fundamental representation of the group and obey

$$[T^a, T^b] = i f_{abc} T^c \tag{47}$$

With the structure constants f_{abc} one constructs the generators of the adjoint representation $T^a_{bc} = i f_{abc}$.

$\mathcal{L}$ has a local gauge invariance. If $G(x)$ is a group matrix, the transformation

$$\psi(x) \to G(x)\psi(x) \qquad W_\mu(x) \to G(x) W_\mu(x) G(x)^{-1} + \frac{i}{g} \partial_\mu G(x) G(x)^{-1} \tag{48}$$

leaves $\mathcal{L}$ invariant. This symmetry is crucial to remove two of the four degrees of freedom in the field W_μ, ($\mu = 0, 1, 2, 3$).

Exercise.- Write the lagrangian (45) in term of group invariants (traces).

3.2 Gauge fixing

To quantize the theory one must select a gauge. The bilinear part in the W_μ field is

$$\frac{1}{2} W^a_\mu (k^2 g^{\mu\nu} - k^\mu k^\nu) W^a_\nu \equiv \frac{1}{2} W^a_\mu M^{\mu\nu} W^a_\nu; \tag{49}$$

$M^{\mu\nu}$ cannot be inverted to find the propagator. The way out is to add the piece

$$\frac{-1}{2\xi} (\partial^\mu W^a_\mu)^2. \tag{50}$$

Then

$$M^{\mu\nu} = k^2 g^{\mu\nu} - (1 - \frac{1}{\xi}) k^\mu k^\nu \qquad (M^{-1})^{\mu\nu} = \frac{g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2 + i\epsilon}. \tag{51}$$

We can now write Feynman diagrams. The added term (50) breaks the local gauge symmetry which, generally speaking, is only recovered for S-matrix elements.

Exercise.- Local symmetries: Try to repeat the arguments leading to the derivation of a Noether current (4), but now considering ϵ_a as function of x, i.e. $\epsilon_a(x)$. Obviously we cannot factor out ϵ_a now. Check that if we replace ∂_μ by D_μ with a gauge connection we can remove the offending term by choosing an appropriate transformation for the gauge field.

Exercise.- Determine the transformation of D_μ and $F_{\mu\nu}$ in a non-abelian gauge theory under a gauge transformation $G(x)$.

How can be sure, though, that by adding the gauge breaking term we are not unduly forcing the theory? We shall present here a proof for the somewhat simpler case of an abelian gauge theory and then comment the differences in the non-abelian case

The generating functional is

$$Z[J] = N \int [dA_\mu] \exp iS[J] \qquad S[J] = \int d^4x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu) \tag{52}$$

If we stick to an abelian theory (e.g. electromagnetism), $F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$. We decompose

$$A_\mu^T = P_{\mu\nu}^T A_\nu, \; A_\mu^L = P_{\mu\nu}^L A_\nu \tag{53}$$

with (in momentum space)

$$P_{\mu\nu}^T = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \; P_{\mu\nu}^L = \frac{k_\mu k_\nu}{k^2}. \tag{54}$$

Then

$$S = -\int d^4x \frac{1}{2} (\partial_\mu A_\nu^T)^2 \tag{55}$$

and therefore the integral over A_μ^L diverges. Note that a gauge transformation $A_\mu \to A'_\mu = A_\mu - \partial_\mu \theta$ changes A_μ^L but not A_μ^T

To avoid this the functional integral must only include one element of each gauge orbit. We want (for instance) to select those representants of the gauge equivalence class that fulfill some condition. For instance

$$F = \partial^\mu A_\mu - f(x) = 0. \tag{56}$$

Under a gauge transformation $F \to F - \Box\theta$. Therefore

$$\int [dA^T] = \int [dA^T][d\theta]\mathrm{det}\frac{\delta F}{\delta\theta}\delta(F) \equiv \int [dA_\mu]\mathrm{det}(-\Box)\delta(\partial^\mu A_\mu - f). \tag{57}$$

We can write Z then as

$$Z = N' \int [df][dA_\mu]\mathrm{det}(-\Box)\delta(\partial^\mu A_\mu - f)G[f]\exp iS, \tag{58}$$

where $G[f]$ is an arbitrary functional of f. For instance

$$G[f] = \exp -\frac{i}{2a}\int d^4x f^2. \tag{59}$$

Then

$$Z = N' \int [dA_\mu]\mathrm{det}(-\Box)\ \exp i(S - \frac{1}{2a}\int d^4x(\partial^\mu A_\mu)^2). \tag{60}$$

Note that $\mathrm{det}(-\Box)$ is field-independent.

For a non-abelian gauge theory things are pretty much the same, but now

$$F^a = \partial^\mu A^a_\mu - f^a(x) = 0. \tag{61}$$

$$\delta F^a = -(\partial^\mu D_\mu)^{ab}\theta^b. \tag{62}$$

and

$$Z = N' \int [dA^a_\mu]\mathrm{det}(-(\partial^\mu D_\mu)^{ab})\ \exp i(S - \frac{1}{2a}\int d^4x(\partial^\mu A^a_\mu)^2). \tag{63}$$

The last step is to introduce some anticommuting variables, the Faddeev-Popov fields that allow us to exponentiate the determinant

$$Z = N' \int [dA^a_\mu]\ \exp i(S - \frac{1}{2a}\int d^4x(\partial^\mu A^a_\mu)^2 + \int d^4x\partial^\mu\bar{c}^a D^{ab}_\mu c^b). \tag{64}$$

Faddeev-Popov fields have a bosonic lagrangian but Fermi statistics —they are unphysical ghost fields. In a while we shall discuss the implications of Faddev Popov ghosts.

3.3 Feynman rules

We shall abandon at this point the general reference to non-abelian gauge theories and go to the specific case of QCD and particularize (45) to $\mathcal{L}_{\mathcal{QCD}}$. The gauge group is $SU(3)$. The generators in the fundamental representation are the Gell-Mann 3×3 matrices, but needless to say that the *colour* group $SU(3)$ has nothing to do with the *flavour* $SU(3)$ group of Gell-Mann. Both groups commute.

$$T^a_{\alpha\beta} = \frac{\lambda^a_{\alpha\beta}}{2} \qquad [T^a, T^b] = if_{abc}T^c \tag{65}$$

The indices $a, b, c, ..$ run over the number of generators of $SU(3)$, namely from 1 to 8. With the structure constants f_{abc} one constructs the generators of the adjoint representation $T^a_{bc} = if_{abc}$.

Some useful relations are

$$\sum_{a\beta} T^a_{\alpha\beta} T^a_{\beta\gamma} = C_F \delta_{\alpha\gamma}, \qquad \sum_{ac} T^a_{bc} T^a_{cd} = C_A \delta_{bd}, \qquad \mathrm{Tr}(T^a T^b) = T\delta_{ab}. \tag{66}$$

In QCD $C_F = 4/3$, $C_A = 3$ and for generators in the fundamental representation $T = T_F = 1/2$, while in the adjoint representation $T = T_A = 3$.

Thus the contents of the QCD lagrangian is a number n of flavours or quark types $i(i = u$, d, s, etc.), each appearing in three different colours labelled by greek letters $\alpha, \beta, ...$; eight gluons and the corresponding Faddeev-Popov ghosts.

The corresponding Feynman rules are provided in a separate figure (with some obvious changes of notation; borrowed from www.phys.psu.edu/ cteq/handbook/v1.1/handbook.ps.gz)

Exercise.- Verify that the Feynman rule for the three-gluon vertex complies with Bose symmetry.

3.4 Ghosts and unitarity

$\mathcal{L}_{QCD}$ has several types of interaction vertices. Let us now consider the process $\bar{q}q \to gg$. At tree level the appropriate diagrams are shown in figure 4

Diagram (c) is absent in the analogous QED process $e^+e^- \to \gamma\gamma$. Due to this diagram it turns out that the above process has a bad high-energy behaviour when we sum over final state polarizations covariantly

$$P = \frac{1}{2} g_{\mu_1\nu_1} g_{\mu_2\nu_2} J^{\mu_1\mu_2} (J^{\nu_1\nu_2})^\dagger, \tag{67}$$

but it is just fine if we include transverse polarizations only

$$P = \frac{1}{2} \tau_{\mu_1\nu_1} \tau_{\mu_2\nu_2} J^{\mu_1\mu_2} (J^{\nu_1\nu_2})^\dagger. \tag{68}$$

The tensors $g_{\mu\nu}$ and $\tau_{\mu\nu}$ are obtained by summing the polarization vectors $\epsilon_\mu(\sigma)\epsilon_\nu(\sigma)$ over all polarizations or over physical ones only, respectively. If we insist in keeping a covariant formalism in which we sum over all four polarizations something must cancel this undesirable high-energy behaviour. This what is accomplished by including the ghost fields. The fields

(a) Propagators: Gluon, quark, and ghost lines of momentum k

ν, a  μ, b $\quad i\frac{\delta_{ba}}{k^2+i\epsilon}[-g^{\mu\nu} + \left(1 - \frac{1}{\lambda}\right)\frac{k^\mu k^\nu}{k^2+i\epsilon}]$ covariant gauge

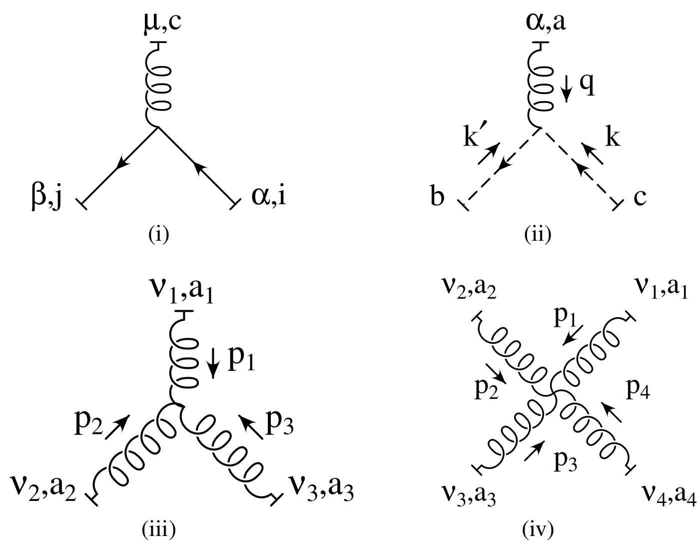
$i\frac{\delta_{ba}}{k^2+i\epsilon}[-g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{n\cdot k} - n^2\frac{k^\mu k^\nu}{(n\cdot k)^2}]$ physical gauge

$k \rightarrow$

α, i  β, j $\quad i\frac{\delta_{ij}}{k^2-m^2+i\epsilon}[\not{k} + m]_{\beta\alpha}$

a  b $\quad i\frac{\delta_{ba}}{k^2+i\epsilon}$

(b) Vertices (all momenta defined to flow in)



(i) $-ig[T_c^{(F)}]_{ji}[\gamma_\mu]_{\beta\alpha}$

(ii) $gC_{abc}k'_\alpha$

(iii) $-gC_{a_1a_2a_3}[g^{\nu_1\nu_2}(p_1 - p_2)^{\nu_3} + g^{\nu_2\nu_3}(p_2 - p_3)^{\nu_1} + g^{\nu_3\nu_1}(p_3 - p_1)^{\nu_2}]$

(iv)
$$
\begin{aligned}
-ig^2[\quad & C_{ea_1a_2}C_{ea_3a_4}(g^{\nu_1\nu_3}g^{\nu_2\nu_4} - g^{\nu_1\nu_4}g^{\nu_2\nu_3}) \\
+ \quad & C_{ea_1a_3}C_{ea_4a_2}(g^{\nu_1\nu_4}g^{\nu_3\nu_2} - g^{\nu_1\nu_2}g^{\nu_3\nu_4}) \\
+ \quad & C_{ea_1a_4}C_{ea_2a_3}(g^{\nu_1\nu_2}g^{\nu_4\nu_3} - g^{\nu_1\nu_3}g^{\nu_4\nu_2})]
\end{aligned}
$$

Figure 1: Perturbation theory rules for QCD.

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Figure 3: Relevant Feynman rules for QCD

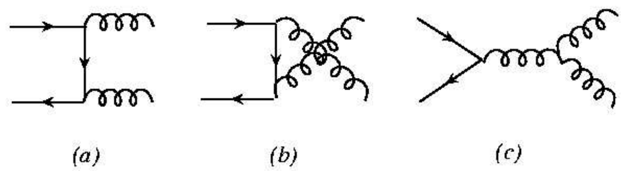

Figure 4: The relevant diagrams for the $\bar{q}q \to gg$ process.

c^a have boson-like couplings, but are defined to have Fermi statistics. They contribute with a (-1) factor to the cross-section. They are not required in abelian theories like QED, but are crucial in QCD.

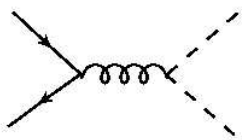

Figure 5: Ghost contribution to $\bar{q}q \to gg$ cross-section.

If all polarizations, physical and unphysical, are summed over one must accept that c states can be produced even if they are ghost states with unphysical statistics. The contribution from the piece we have added to the lagrangian is just right to reproduce the same results we would get keeping the physical polarizations only. In practice, in internal loops we have really no choice but to keep the covariant sum over polarizations and ghosts have to be included there.

Exercise.- Using a decomposition of the photon (or gluon for that matter) field in terms of creation and annihilation operators of a given frequency, write explicitly the four polarization vectors ϵ^μ in Lorenz gauge (note that the Lorenz gauge is Lorentz invariant...). Identify the physical ones.

Exercise.- Find the sum over physical states (i.e. over transverse states in the case of massless gauge particles) of the polarization vectors $\tau^{\mu\nu} = \sum_\sigma \epsilon^\mu(k,\sigma)\epsilon^{\nu *}(k,\sigma)$

Exercise.- What do you think that a "physical" gauge might be? Justify the form of the propagator given in the Feynman rule figure for such a gauge fixing

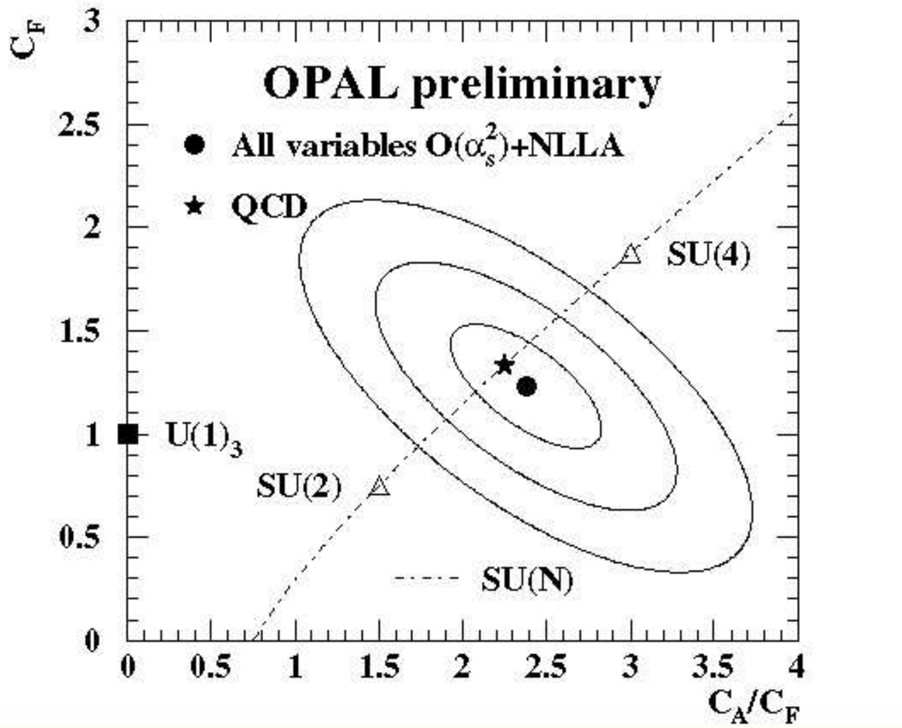

Figure 6: Measuring the group factors (from the OPAL collaboration at LEP).

The group factors can somehow be "measured". For instance the factors C_F, C_A, T_F and T_A appear in jet counting rules (jets are bunches of particles originated from a quark or a gluon, that somehow remember the initial direction of propagation, along the hadronization process, before detection). For instance, the two-jet total cross-section for $e^+e^- \to \bar{q}q$ is proportional to N_c, while the total jet 3 jet cross-section (corresponding to $e^+e^- \to \bar{q}qg$ is proportional to $\alpha_s \times N_c \times C_F$. The agreement between QCD and LEP data is perfect.

Exercise.- Construct observables at tree level for the $e^+e^- \to hadrons$ process with 2, 3, ... hadronic jets in the final state and discuss how these could be use to determine the group C_A, C_F, etc.