4 Global symmetries

In addition to the *local* gauge symmetry we also have exact or approximate *global* symmetries in $\mathcal{L}_{QCD}$. The lagrangian is invariant under the global transformation

$$\psi(x) \rightarrow \exp(-i\theta I)\psi(x) \tag{69}$$

leading to baryon number conservation

$$B = \int d^3x J_0. \tag{70}$$

If all quark masses are equal there are additional symmetries and conserved currents

$$\psi(x) \rightarrow \exp(-i\theta^a T^a)\psi(x) \qquad J_\mu = \bar{\psi}\gamma_\mu T^a \psi. \tag{71}$$

ψ now represents a column vector containing all flavours and T^a is an $SU(N_f)$ generator. Furthermore, if all quark masses vanish $\mathcal{L}_{QCD}$ is also invariant under

$$\psi(x) \rightarrow \exp(-i\theta^a T^a \gamma_5)\psi(x). \tag{72}$$

The currents

$$J_\mu = \bar{\psi}\gamma_\mu\gamma_5 T^a \psi(x) \tag{73}$$

are conserved. In the real world quark masses are not equal, let alone zero. The two latter symmetries are only approximate and this only for light quarks. Therefore the hadronic world is *approximately* invariant under $U(1) \times SU(3)_V \times SU(3)_A$. $U(1)$ is always exact and $SU(3)_V$ is nothing but the vintage $SU(3)_F$ of Gell-Mann, which led to the quark model thirty years ago.

It is perhaps more transparent to write the symmetry group (in the chiral $m_q = 0$ limit) as $SU(3)_L \times SU(3)_R$ since in this limit there is no mixing of left and right degrees of freedom. There is agood reason to do so. While $SU(3)_V$ is indeed a group, $SU(3)_A$ is not. Mathematically it is called a group coset. While it is obvious that

$$[Q_L^a, Q_L^b] = if^{abc}Q_L^c \qquad [Q_R^a, Q_R^b] = if^{abc}Q_R^c \tag{74}$$

it is also true that with $Q_V = Q_R + Q_L$, the commutation relations close, this is not so for $Q_A = Q_R - Q_L$; the commutator of two axial charges is again a vector charge.

$\mathcal{L}_{QCD}$ is also invariant under

$$\psi(x) \rightarrow \exp(-i\theta I\gamma_5)\psi(x) \tag{75}$$

The 'conserved' current is

$$J^5_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi. \tag{76}$$

However, when one computes Green functions with insertions of the divergence of the above current, $\partial^\mu J^5_\mu$, one gets non-zero answers. The culprit is the triangle diagram, the 'anomaly' (the same that we met in π^0 decay, but with the two external photons replaced by gluons). In fact, a careful calculation shows that while the above axial current is conserved at tree level, quantum corrections spoil that conservation and, in fact,

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = \frac{g^2}{4\pi^2}\frac{N_f}{8}\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta}. \tag{77}$$

We shall leave a detailed discussion on this point for the next section. The key point is that there is no way of estimating the divergent momentum integral that appears in the evaluation of fig. 4 without breaking the $U(1)_A$ symmetry, and this is not a point of mathematical finesse; it has far reaching consequences. The r.h.s. of the previous equation is itself a total divergence $\partial^\mu K_\mu$. Then the charge Q_5 verifies

$$\dot{Q}_5 = \int d^3x\partial_0 J^0_5 = \int d^3x\partial_\mu K^\mu - \int d^3x\partial_i J^i_5 \tag{78}$$

In perturbation theory all fields are small perturbations from the vacuum; they decay fast enough to infinity to be able to neglect all boundary terms in the integrals. By Gauss theorem the second integral on the r.h.s. drops and

$$Q_5(t=+\infty) - Q_5(t=-\infty) = \int d^4x\partial_\mu K^\mu = 0 \tag{79}$$

In a non-abelian gauge theory such as QCD there are, however, some non-perturbative gauge configurations (i.e. configurations which are not a superposition of states with a finite number of quarks and gluons) that do not possess the nice long distance behaviour that is required for the boundary term to vanish. These make $\dot{Q}_5 \neq 0$. A conserved charge can still be defined by the integral of $J^5_0 - K_0$, but it is not gauge invariant (the divergence of K_μ is invariant, but not K_μ itself). There is no way to have a conserved, gauge invariant axial charge in QCD. $U(1)_A$ is not a symmetry of the theory.

These non-perturbative contributions are labelled by the Pontryaguin index

$$n = \frac{1}{64\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^{\mu\nu a} \tag{80}$$

For classical solutions of finite energy, n is an integer.

Exercise.- Verify that $\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta}$ is the divergence of the term

$$\epsilon^{\mu\nu\alpha\beta}(W^a_\nu\partial_\rho W^a_\sigma + \frac{g}{3}f^{abc}W^a_\nu W^b_\rho W^c_\sigma) \tag{81}$$

4.1 Realization of global symmetries

As we just discussed, in the chiral limit QCD has a global symmetry group (in the case of three massless flavours) $SU(3)_V \times SU(3)_A \times U(1)$. The existence of this large symmetry group must have far-reaching consequences. We know that invariances of the hamiltonian reflect in a degeneration of the spectrum of particles. This is indeed the case, the subgroup $SU(3)_V$ is nothing but the $SU(3)_F$ invariance —natural extension of isospin— that was introduced by Gell-Mann and others. Their presence gets an immediate translation in hadron octets, decuplets, etc. $U(1)$ is nothing but baryon number; baryon number conservation is automatically implemented in such a theory. There remains another "group", namely $SU(3)_A$. Their presence does not get reflected in yet another set of multiplets. The reason is that unlike in quantum mechanics, in field theory the vacuum is highly non-trivial. Thus, given the vacuum state $|0\rangle$ and a conserved charge Q, two situations may happen

$$Q|0\rangle = 0 \quad \text{or} \quad Q|0\rangle \neq 0 \tag{82}$$

In the first case, the vacuum is invariant under the action of the symmetry that Q represents. Then, so is all the Fock space and multiplets appear

$$|n\rangle = a_n^\dagger|0\rangle \qquad |\tilde{n}\rangle \equiv e^{-iQ}a_n^\dagger e^{iQ}|0\rangle \tag{83}$$

and $E_n = E_{\tilde{n}}$. The symmetry is then realized a la Weyl — the standard way.

The other possibility is of course that $Q|0\rangle = |\alpha\rangle$ where the state $|\alpha\rangle$ is degenerated in energy with the vacuum, but it is not the vacuum itself. If instead of a single generator we have a set of generators forming a Lie algebra, Q^a, corresponding to a given symmetry group G, if two generators leave the vacuum invariant, so does the commutator,

$$[Q^a, Q^b]|0\rangle = 0 \Rightarrow if^{abc}Q^c|0\rangle = 0 \tag{84}$$

therefore the set of generators leaving the vacuum invariant form a subgroup H of G. This is called the unbroken subgroup. In the case of QCD it is $SU(3)_F$.

The generators of G that do not belong to H do not form a group, in general, but a coset $G\backslash H$. The set of states $|\alpha^a\rangle$, where a belongs to one of the broken directions form obviously a multiplet of the unbroken group (Weyl). At zero momentum they describe a multiplet of massless particles. These particles are called Goldstone bosons and the symmetry —we say— is realized a la Goldstone. In QCD, there is an obvious set of candidates, the octet of pseudoscalar mesons 0^-. We believe that in the chiral limit this octet would be exactly massless, the fact that they are pseudoscalar shows that indeed the axial $SU(3)_A$ is the broken "group".

A convenient characterization of the Goldstone phenomenon is provided by the simpler example of a scalar theory with lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi), \tag{85}$$

where ϕ describes a vector invariant under $O(N)$ rotations. As a potential we take

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - \frac{m^2}{\lambda})^2 - \frac{m^4}{4\lambda}. \tag{86}$$

Note that

$$\frac{\partial^2 V}{\partial\phi^2}|_{\phi=0} = m^2. \tag{87}$$

The vacuum energy is clearly lower around the minima described by $\phi \simeq v \equiv m/\sqrt{\lambda}$; It is therefore natural to expect that $\langle\phi\rangle = v \neq 0$. Sinve $v \neq 0$ it must necessarily point somewhere in $O(N)$ space. Therefore for some a, $T^a v \neq 0$ and the direction labelled by a would be a broken one.

Let us now see the appeareance of Goldstone bosons. Since $\mathcal{L}$ is invariant under $\delta\phi = -i\epsilon^a T^a\phi$, it then follows that

$$0 = \frac{\partial V}{\partial\phi}\epsilon^a T^a\phi, \tag{88}$$

differentiating once more we have

$$0 = \frac{\partial^2 V}{\partial\phi^2}T^a\phi + \frac{\partial V}{\partial\phi}T^a. \tag{89}$$

If we place ourselves at the minimum of the potential,

$$\frac{\partial^2 V}{\partial\phi^2}|_{\phi=v}T^a v = 0 \tag{90}$$

Then, if $T^a v = 0$ (unbroken direction), $\frac{\partial^2 V}{\partial\phi^2}|_{\phi=v}$ is arbitrary.

If, on the contrary, $T^a v \neq 0$ (broken direction), $T^a v$ is an eigenvector of $\frac{\partial^2 V}{\partial\phi^2}|_{\phi=v}$ with zero eigenvalue. Since this hessian gives the proper modes (i.e. masses), it implies the existence of a zero mode, i.e. a massless particle in the spectrum.

Let us see how the two discussions, the "classical", in terms of the potential and eigenmodes, and the "quantum", in terms of charges, match. We note that

$$-iT^a v = \langle 0| - iT^a\phi|0\rangle = \langle 0|\delta\phi|0\rangle \sim \langle 0|[Q^a, \phi]|0\rangle. \tag{91}$$

If $T^a v \neq 0$ (broken direction), necessarily $Q^a|0\rangle \neq 0$. Reciprocally, if $Q^a|0\rangle = 0$ then $T^a v = 0$

Exercise.- Would it be possible to have $Q^a|0\rangle \neq 0$, but $T^a v = 0$?
Exercise.- Discuss the spontaneous breaking phenomenon in an quantum ferromagnet in 3 dimensions. Identify symmetry and the broken and unbroken groups. What is $Q|0\rangle$ in this case? Take as hamiltonian of the system

$$H = J \sum_{<i,j>} \vec{S}_i \vec{S}_j, \tag{92}$$

with $s = 1$. The sum extends over nearest-neighbours only.

4.2 Goldstone modes and Goldstone particles

It is useful to discuss this issue in a particular example, such as the Heisenberg model just introduced. Let us assume we are dealing with $s = 1/2$ spins. The invariance group is $SU(2)$.

For $J < 0$ the energy is minimized if the system is fully magnetized, i.e. $\langle S_i^3 \rangle = \pm 1/2$ for all i. For $J > 0$ the minimum energy is attained if

$$\langle S_i^3 \rangle = \pm \frac{1}{2}(-1)^{x_i + y_i + z_i}. \tag{93}$$

In the latter case the system presents an antiferromagnetic, or Néel, state. Of course the choice of the third direction (realization of the vacuum state) is totally arbitrary, but once this is selected, it is clear that the vacuum $|0\rangle$ is no longer $SU(2)$ invariant. For instance, quantizing in the third direction, the Néel state is invariant only under tranformations generated by $S^3 = \sum_i S_i^3$.

In either case, a transformation of the form

$$\xi = \exp(i \sum_{a=1,2} \sum_j \phi_j^a S_j^a), \tag{94}$$

with $\phi_j = \phi$ leaves the energy of the vacumm state unchanged. However, which would be the energy of such a transformed state if ϕ_j is not constant through the lattice? Elementary considerations (and an explicit calculation) shows that such an energy will be of the form

$$|J| \sum_{a=1,2} \sum_{<i,j>} (\phi_i^a - \phi_j^a)^2 \tag{95}$$

with a coefficient in front of order 1. We can write an *effective* hamiltonian appropriate to describe the energy of such excitations

$$|J| \int d^3x (\nabla\phi)^2(x) = -|J| \int d^3x \phi^*(x) \nabla^2 \phi(x), \tag{96}$$

where $\phi = \phi^1 + i\phi^2$.

We can now attempt to write a lagrangian to describe the time evolution of this system. One possibility would be to consider a first order lagrangian

$$L = i\phi^* \partial_t \phi + \frac{1}{2}|J| a^2 \phi^* \nabla^2 \phi, \tag{97}$$

(a is the lattice spacing) whose e-o-m is just the Schrödinger equation

$$i\partial_t \phi = -\frac{1}{2}|J| a^2 \nabla^2 \phi. \tag{98}$$

This leads to a non-relativistic dispersion relation $w \sim k^2$ and the complex field ϕ represents em one particle of mass $m \sim 1/a^2|J|$. This excitation is called a magnon.

Another alternative is to use a second-order lagrangian. Then the appropriate lagrangian would be

$$L = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}c^2 (\nabla\phi)^2, \tag{99}$$

with c being a constant related to $|J|$. This possibility leads to the relativistic-looking wave equation

$$(\partial_t^2 - c^2 \nabla^2)\phi = 0 \tag{100}$$

and to a dispersion relation of the form $w \sim k$, characteristic of *massless* relativistic particles. So only in this case the excitations describe two massless (Goldstone) bosons. They are called spin waves in solid state physics.

Exercise.- Show that with the second order lagrangian

$$L = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}c^2 (\nabla\phi)^2, \tag{101}$$

the energy is still $c^2(\nabla\phi)^2$.

Exercise.- Show that by defining $U = \xi S^3 \xi^\dagger$, the rotational symmetry is linearly realized at the level of the U matrices. Show that the operator $\mathrm{Tr}\nabla U \nabla U$ has the right symmetry properties to describe the energy of arbitrarily large Goldstone mode excitations, and that the term indicated in the previous paragraph is the one with the lowest dimension and the required properties.

As inferred from the previous exercise, we can write the energy of the excitations in a manifestly group invariant form using the matrix U. We can also couple a external magnetic field

B in the third direction to the total magnetization by adding the term $\mathrm{Tr}UB$. It is trivial to see that the derivative of the partition function w.r.t. B just gives $\langle S^3 \rangle$. Those of you familiar with chiral lagrangians will undoubtedly notice the similarity.

What about time derivatives? An obvious choice would be to include a term $(\partial_t U)^2$, that is manifestly invariant. This would of course lead to a relativistic wave equation. How to get a first order lagrangian? A possible term would be $\mathrm{Tr}S^3 \xi \partial_t \xi^\dagger$, but this is apparently not expressable only in terms of the matrix U. Well, it actually is, because the term

$$\int_0^1 d\lambda \epsilon^{\mu\nu} \mathrm{Tr}[U \partial_\mu U \partial_\nu U] \tag{102}$$

with the $U(0) = U$ boundary condition, does the trick for us.

Exercise.- Prove it!

Thus we have the two possibilities. It goes beyond the scope of these notes to see that antiferromagnets are indeed described by a second order-lagrangian, while ferromagnets are described by a first order one. Therefore, antiferromagnets in the broken phase contain massles Goldstone bosons in a number equal to dim G - dim H, while ferromagnets have massive Goldstone modes in a number exactly half the previous one. In conclusion, the Goldstone theorem as is usually stated requieres relativistic invariance.