5 The axial anomaly

In this section we want to show that the axial $U(1)_A$ current is indeed not a symmetry of QCD. We shall do so by computing explicitly the decay $\pi^0 \to \gamma\gamma$ (forbidden in the chiral limit if the axial $U(1)$ current is conserved, as we shall see). A number of new techniques will be introduced. The coupling between quarks and photons is described by the usual piece in quantum electrodynamics: $-eq_i\bar{\psi}_i \not{A}\psi_i$.

We note that the neutral pion is one of the Goldstone bosons of $SU(3)_A$.It corresponds to the broken generator pointing in the λ^3 direction of $SU(3)$. Therefore it must have non-zero overlap with $Q_5^3|0\rangle$. Armed with this knowledge plus covariance arguments

$$0 \neq \langle 0|\bar{\psi}\gamma^\mu\gamma^5\frac{\lambda^3}{2}\psi(0)|\pi(p)\rangle = ip^\mu f_\pi \;\Rightarrow\; \langle 0|\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\frac{\lambda^3}{2}\psi)(0)|\pi(\vec{p})\rangle = f_\pi m_\pi^2. \tag{103}$$

f_π is the pion desintegration constant, roughly equal to 90 MeV.

At this point we need to recall a bit of field theory. If the pion would be described by an elementary field

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}(a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}), \tag{104}$$

and

$$\langle 0|\phi(x)|\vec{p}\rangle = e^{-ipx}. \tag{105}$$

We have used that $\langle\vec{k}|k'\rangle = 2k_0(2\pi)^3\delta(\vec{k}-\vec{k}')$. That means that we can assume that

$$\phi_\pi(x) \equiv \frac{1}{m_\pi^2 f_\pi}\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\frac{\lambda^3}{2}\psi)(x) \tag{106}$$

can be used as interpolating field for the pion.

The last bit of theory we need is the reduction formula. This allows us to go from a Green function to an S matrix element. We shall not provide a justification here but just state the result. If we have a 4-point funtion, with external momenta p_i, in the limit $p_i^2 \to m_i^2$,

$$\langle 0|S|\vec{p_1},..\vec{p_4}\rangle = (-i)^4\prod_i(p_i^2 - m_i^2)\int\prod_i(d^4x_i e^{ip_ix_i})\langle 0|T(\phi(x_1)...\phi(x_4)|0\rangle. \tag{107}$$

In fact the result can be easily understood by considering that by removing the external propagators then one goes from full to amputated diagrams, the latter corresponding to S-matrix elements.

Let us now apply the following considerations to the S-matrix element $\langle 2\gamma|S|\pi(\vec{p})\rangle$. The previous considerations allow us to write this as

$$\lim_{p^2\to m_\pi^2}(p^2-m_\pi^2)\frac{1}{m_\pi^2 f_\pi}\langle 2\gamma|S\partial_\mu(\bar\psi\gamma^\mu\gamma^5\frac{\lambda^3}{2}\psi)(p)|0\rangle. \tag{108}$$

The momentum argument indicates the Fourier transform, namely $O(p)=\int d^4x\exp(ipx)O(x)$. This can be easily computed in perturbation theory. The S-matrix evolution operator contains all of QCD strong interactions, $S=\exp i\int\mathcal{L}^{int}_{QCD}$, electromagnetic interactions are neglected, except for the contribution with the external photons. The leading contribution in QCD perturbation theory is provided by the triangle diagram depicted in fig. 2. However, it is clear that this diagram does not have the right pole structure to cancel the prefactor in (108). To get around this problem we assume that the S matrix element we are interested can be extended of-shell and that the continuation is smooth. This is called the soft-pion limit. Then

$$\langle 2\gamma|S|\pi(\vec{p})\rangle\simeq\frac{1}{f_\pi}\langle 2\gamma|S\partial_\mu(\bar\psi\gamma^\mu\gamma^5\frac{\lambda^3}{2}\psi)(p)|0\rangle. \tag{109}$$

An explicit calculation, using the usual Feynman rules and understanding the new vertex introduced by the operator $\bar\psi\gamma^\mu\gamma^5\frac{\lambda^3}{2}\psi$ gives for one of the two diagrams (and for waht concerns the integration over the space-time degrees of freedom only)

$$\frac{-1}{f_\pi}\int\frac{d^4k}{(2\pi)^4}Tr[\frac{i}{\not k-\not k_1+\not p-m}(-i\not p)\gamma^5\frac{i}{\not k-\not k_1-m}ieq_i\gamma^\mu\frac{i}{\not k-m}ieq_i\gamma^\nu]. \tag{110}$$

Here k_1 is the moment of one of the photons and p the momentum carried by the pion.

Before doing the integral we note that by the equations of motion, naively $\partial_\mu J_5^\mu=0$ (in the chiral limit), so we are allowed to think that this is the final result, and in fact doing formal manipulations in the integral, and using that

$$Tr[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta]=+4i\epsilon^{\mu\nu\alpha\beta} \tag{111}$$

we arrive at

$$\frac{i8e^2q_i^2}{f_\pi}m^2\epsilon^{\alpha\beta\mu\nu}k_{1\alpha}k_{2\beta}\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(k-k_2)^2-m^2][(k-k_1)^2-m^2][k^2-m^2]}. \tag{112}$$

Since on-shell $k_1^2=k_2^2=0$, the largest mass scale in the denominator is p^2, that we take as a free parameter (even though we shall be eventually interested in taking $p^2=0$ because of the soft pion limit, but this is a separate issue). Therefore the integral must behave for small m as $1/p^2$ and we indeed get zero in the chiral limit. A more sophisticated version of this result is the so-called Sutherland-Veltman theorem.

Exercise.- Derive eq. (112)

However, experimentally we have that $\Gamma^{exp} \simeq 7.37 \pm 1.5$ eV. The way out lies in some manipulation we performed that are valid for well defined integrals only, such as shifting the variable of integration. We can fix this by adding a Pauli-Villars field —a heavy fermion with opposite statistics. The contribution of this field will make all fermionic integrals (such as the triangle one) converge, so the formal manipulations are now justified. The heavy "fermion" does not decouple however. Its final contribution will be

$$\frac{-i8e^2 q_i^2}{f_\pi} M^2 \epsilon^{\alpha\beta\mu\nu} k_{1\alpha} k_{2\beta} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k-k_2)^2 - M^2][(k-k_1)^2 - M^2][k^2 - M^2]}. \tag{113}$$

We can clearly neglect k_1 and k_2 in the denominator. Using

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - M^2]^3} = -i \frac{1}{32\pi^2 M^2}, \tag{114}$$

we get after including the contribution of the two quarks with their appropriate charges (i.e. after multiplying by the trace over the internal degrees of freedom),

$$\frac{\alpha}{3\pi f_\pi} \epsilon^{\alpha\beta\mu\nu} k_{1\alpha} k_{2\beta} \epsilon_\mu(k_1, \sigma) \epsilon_\nu(k_2, \sigma'). \tag{115}$$

Finally,

$$\Gamma = \frac{1}{9} \frac{1}{64\pi^3} \frac{\alpha^2}{f_\pi^2} m_\pi^3 = 0.85\text{eV}. \tag{116}$$

The discrepancy is naturally due to colour, that contributes with and additional factor of 9. Note that the same result could have been obtained by assuming that for a single fermion of charge e

$$\partial_\mu J_5^\mu = -\frac{\alpha}{4\pi} F\tilde{F}, \tag{117}$$

where $\tilde{F}^{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu}$, and using the standard QED Feynman rules. The l.h.s. of the above equation is called the anomaly.

Exercise.- Show that the coefficient of the anomaly above is correct.

Exercise.- Compute the anomaly in two space-time dimensions.

The Adler-Bardeen theorem states that what we have just computed remains valid to all orders in perturbation theory, apart from the renormalization of the parameters appearing at lowest order (masses, coupling constants,...). In the case of masses, i.e. away from the chiral limit, the anomaly equation reads

$$\partial_\mu J_5^\mu = 2im\bar{\psi}\gamma^5\psi - \frac{\alpha}{4\pi} F\tilde{F}, \tag{118}$$

for a single fermion of mass m and charge e.

In our case,

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\frac{\lambda^3}{2}\psi) = 2i\bar{\psi}\gamma^5\frac{\lambda^3}{2}\mathcal{M}\psi - \frac{\alpha}{4\pi}Tr[\frac{\lambda^3}{2}Q^2]F\tilde{F}, \tag{119}$$

where Q is the charge matrix in $SU(3)_F$ space and $\mathcal{M}$ is likewise the mass matrix.

To determine the effect of the axial anomaly in different currents is quite straightforward. There is obviously no anomaly in $\pi^0 \to 2g$, but there is one for the $U(1)_A$ current

From the arguments, the true origin of the anomaly should be clear, there is a clash between the symmetries of the classical lagrangian and the regulator that is required to make sense of the divergent integrals in the theory. We have derived the anomaly using a Pauli-Villars regulator, but it can be done by other methods, such as dimensional regularization. This requires a careful treatment of γ^5 away from four dimensions and it is a lot less clear — but obviously correct.

Exercise.- Prove that away from an integer (and even) number of dimensions one cannot define an hermitian operator with the properties $\{\gamma^5, \gamma_\mu\} = 0$, $(\gamma^5)^2 = 1$. Discuss possible ways to proceed in dimensional regularization, where the number of dimensions is non-integer.