6 Regularization and renormalization

Beyond tree level most Feynman diagrams are ultraviolet divergent. Take for instance the first diagram of those contributing at one loop to the gluon propagator (Figure 7)

Figure 7: One loop contributions to the gluon self energy

Neglecting external momenta, the integral over the momenta of the internal particles is of the form

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^\alpha k^\beta}{k^4} = \infty. \tag{120}$$

It should be said right away that, although the power counting looks disastrous, the situation is actually not as bad due to gauge invariance. Indeed, if everything is done in a gauge invariant way, only gauge invariant terms can appear as a result, with a well prescribed, form and this actually reduces the numberof divergent diagrams and, for those thata are still divergent, the degree of the divergence. The integrals diverge no worse than logarithmically.

Still, to make sense of the theory and get a finite result we must introduce a cut-off Λ to regulate the badly behaved integrals and counterterms. A typical method is to perform a subtraction at some $q^2 = -\mu^2$. For instance, for the self-energy of the gluon propagator.

$$\Pi(q^2) - \Pi(-\mu^2) \equiv \Pi_R(q^2) = \text{finite}. \tag{121}$$

Alternatively we can make sense of the integrals using dimensional regularization by continuing the dimensionality from 4 to $n = 4 + 2\epsilon$,

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^nk}{(2\pi)^n}. \tag{122}$$

When doing so, we must be quite careful with the dimensionalities of the fields. In $n = 4+2\epsilon$ dimensions,

$$[W_\mu] = M^{1+\epsilon} \quad [\psi] = M^{\frac{3}{2}+\epsilon} \quad [g] = -\epsilon \qquad etc. \tag{123}$$

Note that the coupling constant is dimensionful, in order to restore the canonical dimensions, it is convenient to take out these dimensions explicitly, this introduces a factor $\mu^{-\epsilon}$ in each integral. The quantity μ is in principle completely arbitrary.

Exercise.- Determine the dimensionality of the gauge-fixing parameter in dimensional regularization.

't Hooft proved that in $n = 4 + 2\epsilon$ dimensions Feynman integrals are meromorphic functions in ϵ, diverging when $\epsilon \to 0$. To renormalize, i.e. to subtract the divergences, we can then subtract just the poles in $1/\epsilon$ (minimal subtraction, MS) or variations thereof such as subtracting also the $\gamma_E - \log 4\pi$ that always accompanies the singularity in $1/\epsilon$ (improved minimal subtraction, $\overline{MS}$).

For instance, for the third diagram in fig. 5, namely the quark contribution to the gluon self-energy, one has the following result after computing the integral in $n = 4+2\epsilon$ dimensions

$$\Pi(q^2) = -\frac{\alpha_s}{6\pi}\delta_{ab}(\frac{1}{\epsilon} + \gamma_E + \log\frac{m^2}{4\pi\mu^2} + F(\frac{q^2}{m^2})). \tag{124}$$

Using the MS and $\overline{MS}$ schemes one gets

$$\Pi_{MS}(q^2) = -\frac{\alpha_s}{6\pi}\delta_{ab}(\gamma_E + \log\frac{m^2}{4\pi\mu^2} + F(\frac{q^2}{m^2})), \qquad \Pi_{\overline{MS}}(q^2) = -\frac{\alpha_s}{6\pi}\delta_{ab}(\log\frac{m^2}{\mu^2} + F(\frac{q^2}{m^2})) \tag{125}$$

μ is the dimensional quantity we have been forced to introduce in dimensional regularization.

Admittedly this looks totally adhoc. It would seem that we can get rid of any divergent integral by just adding the appropriate counterterms. Of course, since these are sort of arbitrary, we would then be able to obtain any result we want. This is not so, of course.

Even admitting that there is a degree of arbitrariness (whe shall see later how this is resolved), it should be clear that this just consists in a subtraction cosntant; the full momentum dependence is unambiguous. In a renormalizable theory, only a finite number of Green functions are divergent and hence the ambiguity lies in a finite number of constants. A renormalizable theory is one in which the necessary counterterms to all orders in perturbation theory are generated by redefining the fields and parametres of the original lagrangian and nothing else, and this is highly non-trivial. QCD is such a theory.

If we redefine

$$g^0\mu^\epsilon \quad = Z_{1YM}Z_{3YM}^{-3/2}g = \tilde{Z}_1\tilde{Z}_3^{-1}Z_{3YM}^{-1/2}g = Z_{1F}Z_{3YM}^{-1/2}Z_{2F}^{-1}g = Z_5^{1/2}Z_{3YM}^{-1}g \equiv Z_g g,$$

$$
\begin{aligned}
W^0_\mu \mu^{-\epsilon} &= Z^{1/2}_{3YM} W_\mu, \\
c^0 \mu^{-\epsilon} &= \tilde{Z}^{1/2}_3 c, \\
\psi^0 \mu^{-\epsilon} &= Z^{1/2}_{2F} \psi,
\end{aligned}
$$

it is possible by a suitable (non-unique) choice of the Z's to make QCD finite. The redefinition of the gauge-fixing parameter has not been included for simplicity, but it is necessary and it involves a new constant Z_6. Note that the so-called 'bare' quantities, labelled with a '0' are the ones having non-canonical dimensions, in particular g_0 has some dimensions. The Z's are all dimensionless. The first string of identities stems from the fact that the different vertices of the theory should have the same coupling in order to keep gauge invariance.

Since all Z's are of the form $Z = 1 + \Delta Z$, with ΔZ beginning at $\mathcal{O}(g^2)$ this amounts to replacing the original $\mathcal{L}_{QCD}$ one started with by $\mathcal{L}_{QCD} + \Delta\mathcal{L}_{QCD}$. $\Delta\mathcal{L}_{QCD}$ contains the necessary counterterms that must be added. Equivalently, we can work directly with the QCD lagrangian with all fields and parametres replaced by the 'bare' ones, defined through eq. (126) Observables will be obtained by the prescription (Λ being the ultraviolet cut-off)

$$
\Gamma(p_i, g, \mu) = \lim_{\Lambda\to\infty} Z^{-m/2}_{3YM} Z^{-k/2}_{2F} \Gamma_0(p_i, g_0, \Lambda). \tag{126}
$$

On the l.h.s. we have renormalized Green functions or amplitudes expressed as a function of renormalized parametres. On the r.h.s we have bare Green functions or amplitudes expressed as a function of bare parametres. In dimensional regularization

$$
\Gamma(p_i, g, \mu) = \lim_{\epsilon\to 0} Z^{-m/2}_{3YM} Z^{-k/2}_{2F} \Gamma_0(p_i, g_0, \epsilon). \tag{127}
$$

μ is the subtraction scale and m and k are the number of gluon and quark external lines, respectively.

Let me stress that the coupling constant appearing on the r.h.s. of the previous expression is dimensionful and this makes up for the non-canonical dimensions of the space-time integrals in order to produce a renormalized Green function with the right dimensions.

The effect of counterterms is to replace the dependence on the cut-off (Λ, ϵ, ...) by a dependence on μ. Unlike in QED where the natural scale is m_e, or the Electroweak theory where the natural scale is M^2_W, there is really no preferred way of choosing the counterterms. The only requirement is the fulfillment of the Ward identities (126. In practice MS and $\overline{MS}$ are the most useful, particularly the latter that seems to lead to perturbative series with a faster convergence. In the MS scheme all renormalization constants are just poles in ϵ

$$Z = 1 + \frac{\alpha_s}{\pi}\frac{a}{\epsilon} + (\frac{\alpha_s}{\pi})^2(\frac{b}{\epsilon^2} + \frac{c}{\epsilon}) + \ldots = 1 + \frac{\alpha_s^0}{\pi}\mu^{2\epsilon}\frac{a}{\epsilon} + \ldots \tag{128}$$

To make sense of the last identity one has to recall again that α_s^0 is dimensionful.

6.1 Effective charge in QED

To understand the meaning of renormalization it is useful to focus on a theory that is simpler than QCD, namely QED. In this theory, the only contribution to the photon self-energy is given by the diagram

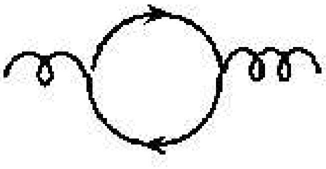

Figure 8: One loop contributions to the photon self energy

whose result, using the Feynman rules of QED defines the polarization tensor

$$-i(k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi(k^2). \tag{129}$$

The fact that this Green function has this peculiar (transverse) structure is again a consequence of gauge invariante. This can be proven with the help of the so-called Ward identities. In this particular case, they state that the gauge boson propagator self-energy is 100% transverse beyond tree level.

Iteration of the self-energy gives (we show only the first term in the series)

$$(-i)\frac{g^{\mu\beta} - \frac{k^\mu k^\beta + \xi k^\mu k^\beta}{k^2}}{k^2 + i\epsilon} + \tag{130}$$

$$(-i)\frac{g^{\mu\nu} - \frac{k^\mu k^\nu + \xi k^\mu k^\nu}{k^2}}{k^2 + i\epsilon} \times (-i)(k^2 g_{\nu\alpha} - k_\nu k_\alpha)\Pi(k^2) \times (-i)\frac{g^{\alpha\beta} - \frac{k^\alpha k^\beta + \xi k^\alpha k^\beta}{k^2}}{k^2 + i\epsilon} + \ldots \tag{131}$$

The longitudinal parts of the propagator cancel when contracted with the self-energy. We need to keep track of the pieces in $g^{\mu\beta}$. We get

$$\frac{-i}{k^2} g^{\mu\beta}(1 - \Pi(k^2) + \Pi^2(k^2) - \ldots.) = -i g^{\mu\beta}\frac{1}{k^2(1 + \Pi(k^2))}. \tag{132}$$

This means in practical terms that we have to deal with an "effective coupling"

$$\alpha \to \alpha(k^2) = \frac{\alpha}{1+\Pi(k^2)} \tag{133}$$

Of course, $\Pi(k^2)$ is divergent and renormalization is required. Recalling the relation between bare and renormalized gauge fields $W^0_\mu = Z^{\frac{1}{2}}_{3YM} W_\mu$ (the procedure is identical in the case of QCD) and working at a fixed order in perturbation theory, defining $\Gamma = \langle TWW \rangle$

$$\Gamma^{(2)} = \Gamma^{(2)}_0 - \delta Z_{3YM} \Gamma^{(0)}_0 \tag{134}$$

so, the physical photon polarization will be (transverse part)

$$(-i)(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2})(1 - \Pi(k^2) - \delta Z_{3YM}) \tag{135}$$

Measuring the effective charge at some particular $k^2 = \nu^2$ determines *in a particular renormalization scheme* that provides a specific value for δZ_{3YM} the physical value for α.

The previous definition of an effective charge does not actually make sense in QCD, because of the presence of additional interaction vertices. In fact the previous quantity, conveniently carried over to QCD, is not gauge independent.

Exercise.- Check the counterterm structure that makes finite the Green function $\Gamma = \langle TWWW \rangle$ in QCD (W_μ is the gluon field). You have to renormalize external fields and the charge, separate the calculation into Feynman diagrams and see what renormalizes what. Get yourself familiar with the way renormalization works. What is the purpose of Z_{1YM}? What does it make finite?

6.2 Ward identities

This is not the place to treat in detail this issue —a key one for understanding the renormalization of gauge theories, but let suffice here to show with an example the enormous number of relation between different Green functions in a gauge theory.

At tree-level in QED, the photon vertex Feynman diagram reads (unesential factors are omitted)

$$\frac{1}{\not p + \not q - m} \gamma_\mu \frac{1}{\not p - m}, \tag{136}$$

where p is the initial moment of the fermion and q the momentum transfer. Contracting with q^μ we get

$$\frac{1}{\not p + \not q - m} \not q \frac{1}{\not p - m} = \frac{1}{\not p - m} - \frac{1}{\not p + \not q - m}. \tag{137}$$

Multiplying by the inverse fermion propagators it is straightforward to prove the following relation among 1PI Green functions

$$q^\mu \Gamma_\mu = ie(\not{q} + \Sigma(p+q) - \Sigma(p)), \tag{138}$$

where we have taken into account that the (dressed) inverse fermion propagator is $\not{p} - m + \Sigma(p)$.