7 Renormalization-group

Note that regulated quantities depend on a cut-off (Λ, ϵ, ...) and that the renormalization of fields and constants through eq. (126) trades the dependence on the cut-off by some scale μ. Yet, physics cannot depend on μ at all since it is a priori completely arbitarary. If you change μ you must change at the same time the value of your renormalized parametres to make up for the difference. A simple way to encode this observation is the following. Let's write

$$\Gamma(p_i, \alpha_s, \mu) = Z_\Gamma(\mu, \epsilon)\Gamma_0(p_i, \alpha_s^0, \epsilon) \tag{139}$$

Γ_0 is obviously independent of the subtraction scale μ. Therefore

$$0 = \mu\frac{d}{d\mu}\Gamma_0 = Z_\Gamma^{-1}(\mu\frac{d}{d\mu} - Z_\Gamma^{-1}\mu\frac{d}{d\mu}Z_\Gamma)\Gamma. \tag{140}$$

(We neglect everywhere the dependence on the gauge parameter, as well as the quark masses. Of course they have to be properly taken into account. Physical on-shell amplitudes are ξ independent.) From eq. (140)

$$(\mu\frac{\partial}{\partial\mu} + \mu\frac{d\alpha_s}{d\mu}\frac{\partial}{\partial\alpha_s} - Z_\Gamma^{-1}\mu\frac{d}{d\mu}Z_\Gamma)\Gamma = 0, \tag{141}$$

or, defining the so-called β-function and the Green function anomalous dimension γ_Γ,

$$\mu\frac{d\alpha_s}{d\mu} = \alpha_s\beta(\alpha_s) \qquad Z_\Gamma^{-1}\mu\frac{d}{d\mu}Z_\Gamma^{-1} = \gamma_\Gamma, \tag{142}$$

$$(\mu\frac{\partial}{\partial\mu} + \beta\alpha_s\frac{\partial}{\partial\alpha_s} - \gamma_\Gamma)\Gamma(p_i, \alpha_s, \mu) = 0. \tag{143}$$

This is called the renormalization-group equation. Let's investigate its consequences. We can always write Γ as a function of dimensionless variables by pulling out μ^D (D: dimensionality of Γ)

$$\Gamma(\lambda p_i, \alpha_s, \mu) = \mu^D F(\frac{\lambda^2 p_i p_j}{\mu^2}, \alpha_s). \tag{144}$$

Hence

$$(\lambda\frac{\partial}{\partial\lambda} + \mu\frac{\partial}{\partial\mu} - D)\Gamma(\lambda p_i, \alpha_s, \mu) = 0. \tag{145}$$

Using the renormalization-group equation we get

$$(-\frac{\partial}{\partial t} + \beta\alpha_s\frac{\partial}{\partial\alpha_s} - \gamma_\Gamma + D)\Gamma(e^t p_i, \alpha_s, \mu) = 0. \tag{146}$$

From subtraction scale independence arguments we have been able to establish an equation concerning the dependence on the external momenta. We can formally solve this equation

$$\Gamma(e^t p_i, \alpha_s(\mu), \mu) = \exp[tD - \int_0^t dt \gamma_\Gamma(\bar{\alpha}_s(t))] \times \Gamma(p_i, \bar{\alpha}_s(t), \mu), \tag{147}$$

$\bar{\alpha}_s(t)$ is just $\alpha_s(e^t\mu)$, i.e. the same coupling constant but renormalized at a different scale.

Exercise.- Verify that (147) is indeed the solution of the differential equation (146).

From eq. (147) we see that when we scale the external momenta in a Green function or amplitude the change is absorbed
(i) in a multiplicative factor that depends on the anomalous dimension as well as the engineering dimension of the amplitude and
(ii) in a redefinition of the coupling $\alpha_s(\mu) \to \alpha_s(e^t\mu)$.
The renormalization-group evolution of the parametres in the theory (in this case exemplified by the coupling constant α_s) governs the scaling behaviour. We have to find which is the evolution of α_s under a change of μ. This is actually a very simple question. We just have to solve the equation

$$\mu \frac{d\alpha_s}{d\mu} = \alpha_s \beta(\alpha_s). \tag{148}$$

Since the μ dependence of α_s is introduced via counterterms, To compute β we have just to find the relevant renormalization constants (see eq. (126)).

7.1 The beta function

This involves in practice computing, for instance the three gluon vertex, choosing a particular renormalization scheme, and selecting the counterterm for g, Z_g, and for the external legs, Z_{3YM} that makes the Green function finite. In the MS scheme the counterterms are a series in ϵ poles. At one loop level there are single poles, at the two-loop level there are single and double poles, etc.

$$\alpha_s = \lim_{\epsilon \to 0} Z_\alpha^{-1} \alpha_s^0 \mu^{2\epsilon} \tag{149}$$

$Z_\alpha = Z_g^2$ is the quantity that is determined in perturbation theory. Acting with $\mu \frac{d}{d\mu}$ on both sides, and recalling that α_s^0 is μ independent we obtain the perturbative expansion for β

$$\alpha_s \beta = - \lim_{\epsilon \to 0} \alpha_s^0 \frac{1}{Z_\alpha^2} \mu \frac{dZ_\alpha}{d\mu} + 2\epsilon Z_\alpha^{-1} \alpha_s^0. \tag{150}$$

At this point one should remember that the $\log \mu$ are in one to one correspondence with the powers of $1/\epsilon$

Of course β is evaluated in perturbation theory and, accordingly, the above differential equation is also solved in perturbation theory. The solution will only make sense as long as the expansion parametre is small. Thus

$$\beta = 2\epsilon + \beta_1(\frac{\alpha_s}{\pi}) + \beta_2(\frac{\alpha_s}{\pi})^2 + ... \tag{151}$$

At one loop the solution of the renormalization-group equation for α will be given by the solution of the differential equation

$$\frac{\pi}{\beta_1}\frac{d\alpha_s}{\alpha_s^2} = \frac{d\mu}{\mu} \tag{152}$$

which is

$$\frac{1}{\alpha_s(e^t\mu)} = \frac{1}{\alpha_s(\mu)} - \frac{\beta_1}{\pi}t \tag{153}$$

or

$$\alpha_s(e^t\mu) = \frac{\alpha_s(\mu)}{1 - \frac{1}{\pi}\beta_1\alpha_s(\mu)t}. \tag{154}$$

In QCD β_1 (the first coefficient of the β- function) is

$$\beta_1 = -\frac{11}{2} + \frac{N_f}{3}, \tag{155}$$

(recall that in QED $\beta_1 = 2/3$). Note that β_1 is negative if $N_f < 16$. If β_1 is negative, at larger momentum transfers, where the relevant scale will be $e^t\mu$ and not μ, α_s will actually decrease. The solution of the renormalization-group equation will actually become better and better at higher energies.

To substantiate these statements we have to go back to the solution of the renormalization-group equation (147) and let us recall its physical contents. Ignoring for a second the overall factor, when we scale up the momenta we have exactly the same amplitude, but renormalized at scale $e^t\mu$, i.e. replacing $\alpha_s(\mu)$ by $\alpha_s(e^t\mu)$. Obviously the scaled up amplitude will correspond to a theory that interacts more weakly. In the $t \to \infty$ limit we will have a free theory. This is called asymptotic freedom and it is one of the characteristic signals of strong interactions. QCD has it, while QED has not.

We learn something else. Let us imagine that for some reason radiative corrections are small for some amplitude with momenta p_i and the coupling constant renormalized at scale μ. (For instance, in QED we may decide to choose the counterterms and *define* the renormalized coupling constant in such a way that the classical formula for Thompson scattering is strictly valid at all orders in perturbation theory for on-shell particles.) The renormalization group tells us is that if we scale the external momenta *and* the renormalization scale in the way prescribed by eq. 52, radiative corrections will also be small for the scaled amplitude. Each physical process has a characteristic scale. Choosing this scale as subtraction point typically optimizes the perturbative series.

Exercise.- If

$$Z_\alpha = 1 + \frac{\alpha_s}{\pi}\frac{a}{\epsilon} + (\frac{\alpha_s}{\pi})^2(\frac{b}{\epsilon^2} + \frac{c}{\epsilon}) + ..., \tag{156}$$

determine β_1 as a function of a.

Exercise.- Show that there is a relation between b and a. Determine β_2.

7.2 Asymptotic freedom

Whenever we speak of 'higher' energies we are implicitly assuming the existence of some characteristic QCD scale. Looking at eq. (148) we note that

$$t = \frac{1}{2}\log\mu^2 = \int \frac{d\alpha_s}{\alpha_s\beta(\alpha_s)} = \psi(\alpha_s) + C. \tag{157}$$

C is an integration constant. Therefore $t-\psi(\alpha_s)$ is a constant of motion along the renormalization-group trajectory. At one loop, we plug β_1 in the previous equation and get

$$\frac{1}{2}\log\mu^2 + \frac{\pi}{\beta_1\alpha_s(\mu)} = C \equiv \frac{1}{2}\log\Lambda^2_{QCD} \Rightarrow \alpha_s(\mu) = \frac{-\pi}{\frac{\beta_1}{2}\log(\mu^2/\Lambda^2_{QCD})}. \tag{158}$$

If we renormalize at scales much larger than Λ_{QCD} the renormalized coupling constant will be small and working at one loop will be justified. The value of α_s, renormalized at the scale $\mu = M_Z \simeq 90$ GeV, has been measured rather precisely in recent years from different LEP observables (such as the ratio between three and two jets we discussed in previous sections). Note that M_Z is much larger than any hadronic scale. At such energies perturbation theory is clearly meaningful. The preferred value (PDG 2002) is $\alpha_s(M_Z) = 0.119 \pm 0.004$.
We expect the coupling constant α_s to be small at LEP energies because $M_Z \gg \Lambda_{QCD}$, but which evidence do we have that it actually runs according to the renormalization group predictions? Well, actually α_s is not small enough for two-loop effects to be completely neglected so to answer this question in a precise way we have to work a bit harder and compute the two-loop β-function. The result is

$$\beta_2 = -\frac{51}{4} + \frac{19}{12}N_f. \tag{159}$$

Incidentally, both β_1 and β_2 are scheme independent (but not β_3 and beyond).

Note that, for $N_f < 8$, β_2 is also negative. Repeating the steps that led to eq. (158) we get at the two-loop level

$$\frac{1}{2}\log\mu^2 + \frac{\pi}{\beta_1\alpha_s(\mu)} + \frac{\beta_2}{\beta_1^2}\log\alpha_s(\mu) + \mathcal{O}(\alpha_s) = \frac{1}{2}\log\Lambda_{QCD}^2, \tag{160}$$

whose solution is

$$\alpha_s(\mu) = \frac{12\pi}{(33-2N_f)\log(\mu^2/\Lambda_{QCD}^2)}\left[1 - 3\frac{153-19N_f}{(33-2N_f)^2}\frac{\log\log(\mu^2/\Lambda_{QCD}^2)}{\frac{1}{2}\log(\mu^2/\Lambda_{QCD}^2)}\right] \tag{161}$$

Working in the MS or $\overline{MS}$ scheme changes the value of $\alpha_s(\mu)$ but the difference is $\mathcal{O}(\alpha_s^2)$. Therefore the numerical value of Λ_{QCD} does actually depend on the renormalization scheme one is using (one does not see this at the one-loop level precision). For instance, at two loops Λ_{MS} and $\Lambda_{\overline{MS}}$ are related through

$$\Lambda_{MS}^2 = \frac{e^{\gamma_E}}{4\pi}\Lambda_{\overline{MS}}^2. \tag{162}$$

Eq. (161) is thus our starting point to check whether the behaviour predicted by the renormalization group has factual support. However, there is one more thing that we have to account for when we choose to work in the MS or $\overline{MS}$ schemes. Decoupling of heavy fermions is not manifest in any of these schemes. In practice one works with the number of quarks that are excited at the energy one is. For instance, at scales well below and well above the charm threshold we have, respectively (at the one-loop level)

$$\alpha_s(\mu) = \frac{12\pi}{27\log(\mu^2/\Lambda^2(3))} \qquad \alpha_s(\mu) = \frac{12\pi}{25\log(\mu^2/\Lambda^2(4))}. \tag{163}$$

The coupling constant is obviously continuous as we cross the threshold, but Λ_{QCD} is not. Demanding continuity on α_s leads to the matching condition

$$\Lambda(4) = \left(\frac{\Lambda(3)}{m_c}\right)^{\frac{2}{25}}\Lambda(3). \tag{164}$$

At two-loops this gets modified to

$$\Lambda(4) = \left(\frac{\Lambda(3)}{m_c}\right)^{\frac{2}{25}}\left(\log\frac{m_c^2}{\Lambda^2(3)}\right)^{-\frac{107}{1875}}\Lambda(3). \tag{165}$$

If one insists in keeping $\Lambda(3)$ beyond the charm threshold the corrections will be large; perturbation theory breaks down.

The two loop evolution of the QCD coupling constant is shown in a separate plot for different values of $\Lambda_{\overline{MS}}$. On top, the values obtained from different experiments at different scales are shown. Asymptotic freedom is very nicely confirmed. We cannot go to energies much lower than those shown because the coupling constant grows very quickly and three-loop effects and beyond are relevant; in fact, perturbation theory becomes meaningless. In the coming sections we will discuss the ways determining α_s that is shown in this plot.

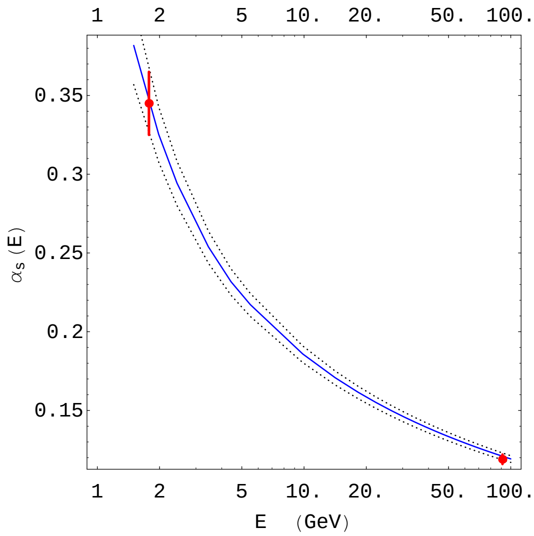

Figure 9: The running of α_s.