9 R_{had}

The observable

$$R_{had} = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{169}$$

is probably the cleanest and simplest observable in QCD. It is fully inclusive and can be computed through a dispersion relation which is symbollically depicted in fig. 11.

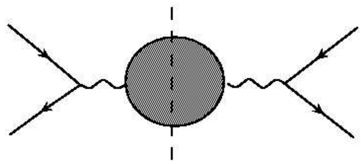

Figure 11: The imaginary part of the self-energy describes the $\gamma^* \to hadrons$ width.

To understand how this comes about we use the by now familiar Feynman rules to compute the cross-section $\sigma(e^+e^- \to hadrons)$.

We note that

$$iM(2\pi)^4\delta^{(4)}(q - P_\Gamma) = \langle\Gamma|S - I|k_1k_2\rangle \tag{170}$$

with $k_1 + k_2 = q$. $S - I$ is obtained by expanding the interaction lagrangian $\exp(i\int d^4x\mathcal{L}_I$. In our case, assuming that we treat the electromagnetic interaction at tree level, taking into account that for the electromagnetic part $\mathcal{L}_I = \sum eQ_iJ^q_\mu A^\mu + eJ^e_\mu A^\mu$, where J^q_μ and J^e_μ are the quark and electron electromagnetic currents, respectively, we obtain after use of Wick's theorem

$$\langle\Gamma|S{-}I|k_1k_2\rangle = -\frac{1}{2}\int d^4xd^4y\, e^2\sum_\Gamma Q_i\langle\Gamma|J^i_\mu(x)|0\rangle\langle 0|T(A^\mu(x)A^\nu(y))|0\rangle\langle 0|J^e_\nu(y)|k_1k_2\rangle. \tag{171}$$

Furthermore

$$\langle\Gamma|J^i_\mu(x)|0\rangle = e^{iP_\Gamma x}\langle\Gamma|J^i_\mu(0)|0\rangle. \tag{172}$$

The charges are explicitly pulled out of the currents. If we assume that the final state is a quark-antiquark with momenta p_1, p_2 state we get for the last matrix element $\bar{u}(p_1)\gamma_\mu v(p_2)$, and likewise we can construct the appropriate matrix element for any perturbative state, but the above expression actually holds for *any* final state, perturbative (quraks, gluons) or not (hadrons).

After some standard manipulations one arrives at

$$\sigma(e^+e^- \to hadrons) = \frac{1}{\sqrt{q^2(q^2-4m_e^2)}} \frac{e^4 \sum Q_i^2}{q^4} L^{\mu\nu}(k_1,k_2)\Pi^{(abs)}_{\mu\nu}(q) \tag{173}$$

with the lepton tensor being

$$L^{\mu\nu}(k_1,k_2) = \frac{1}{2}[(q^\mu q^\nu - q^2 g^{\mu\nu}) - (k_1-k_2)^\mu (k_1-k_2)^\nu] \tag{174}$$

and the hadronic tensor

$$\Pi^{(abs)}_{\mu\nu}(q) = \frac{1}{2}\sum_\Gamma (2\pi)^4 \delta^{(4)}(q-P_\Gamma)\langle 0|J_\mu(0)|\Gamma\rangle\langle\Gamma|J_\nu(0)|0\rangle \tag{175}$$

Exercise.- Derive eq. (173).

We now wish to show that $\Pi^{(abs)}_{\mu\nu}(q)$ is actually the imaginary part of the Green function

$$\frac{1}{2}\int d^4x e^{iqx}\langle 0|T(J_\mu(x)J_\nu(0))|0\rangle. \tag{176}$$

We actually see that

$$\Pi^{(abs)}_{\mu\nu}(q) = \frac{1}{2}\sum_\Gamma \int d^4x e^{ix(q-P_\Gamma)}\langle 0|J_\mu(0)|\Gamma\rangle\langle\Gamma|J_\nu(0)|0\rangle, \tag{177}$$

using a resolution of the identity this equals

$$\frac{1}{2}\int d^4x e^{iqx}\langle 0|J_\mu(x)J_\nu(0)|0\rangle \tag{178}$$

Exercise.- We leave as an exercise to derive the rest of the proof. Hints: (a) write the time-ordered product in term of step functions (b) do the same for Π^* and perform some shifts in the integration variables and use that $\Pi_{\mu\nu}$ is symmetric in $\mu\nu$ (c) use translation operators and recall that physical states have positive energy.

The sum over intermediate states on the r.h.s. of fig. 11 thus runs over all hadronic states. From the previous discussion we see that we can just as well compute this sum using another resolution of the identity—the one that is provided to us by perturbative QCD, i.e. in terms of quarks and gluons.

One finally gets

$$\sigma(e^+e^- \to hadrons) = N_c \frac{4\pi\alpha^2}{3q^2}\sum Q_i^2 \mathrm{Im}\Pi(q^2) \tag{179}$$

The analogous quantity for muons is

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3q^2} \tag{180}$$

Let us see how this works. At lowest order in α_s in the massless limit the quark contribution to the photon self-energy (this actually what the above correlator is) is just (the color factor has been extracted)

$$\Pi(q^2) = \frac{1}{\pi}[\frac{1}{\epsilon} + \gamma_E + \log\frac{-q^2}{4\pi\mu^2} - \frac{5}{3}] \tag{181}$$

whose imaginary part is simply i. One immediately gets that at lowest order $R_{had} = N_c \sum Q_i^2$.

Exercise.- Discuss the relevance of the choice of the cut in the logarithm in the complex plane. How can we guarantee the right sign. Notice that the result is a cross-section, hence positive.

R_{had} has been computed in this way up to third order in α_s, with some results at order α_s^4.

$$R_{had} = R_0[1 + \frac{\alpha_s(q^2)}{\pi} + r_2(\frac{\alpha_s(q^2)}{\pi})^2 + r_3(\frac{\alpha_s(q^2)}{\pi})^3 + \ldots], \tag{182}$$

where q^2 is the momentum transfer (choosen to suppress the $log\frac{q^2}{\mu^2}$, μ being the renormalization scale, and in the $\overline{MS}$ scheme

$$r_2 \simeq 2.0 - 0.12N_f \qquad r_3 = -6.637 - 1.200N_f - 0.005N_f^2 - 1.240\frac{(\sum Q_i)^2}{3\sum Q_i^2}. \tag{183}$$

For $N_f = 5$, $r_2 \simeq 1.4$ and $r_3 \simeq -12.8$. We have seen that $\alpha_s(M_Z) \simeq 0.12$. Then

$$R_{had} = R_0[1 + 0.04 + 0.002 - 0.0008 + \ldots] \tag{184}$$

The convergence of the series does not look bad, but it is not very good either. The values for the coefficients that we have been just quoted correspond to massless quark; they have to be accordingly modified for heavy quarks. When this is done and comparison with experiment is done we can extract a value for $\alpha_s(M_Z)$.
The convergence of the perturbative series is much worse in the MS scheme. It is not clear why it is so good in the $\overline{MS}$, because in Field Theory this is often the case only if one uses a physically motivated scheme (such a subtraction at some energy scale), while the reasons to use the $\overline{MS}$ scheme are of practical order. Be as it may, this is a welcome fact.

R_{had} is an observable so it must be independent of the renormalization scheme. If we work in the MS scheme where the r_i are different something else must change so that the net result is still the same. The thing that changes is of course the coupling constant itself

$$\alpha_s^{\overline{MS}}(\mu) - \alpha_s^{MS}(\mu) = \frac{12\pi}{(33-2N_f)\log(\mu^2/\Lambda^2_{\overline{MS}})} \frac{\log 4\pi - \gamma_E}{\log(\mu^2/\Lambda^2_{\overline{MS}})}, \tag{185}$$

which just makes up for the changes (up to log log terms), i.e.

$$1 + \frac{\alpha_s^{MS}(\mu)}{\pi} + r_2^{MS}(\frac{\alpha_s^{MS}(\mu)}{\pi})^2 + \ldots = 1 + \frac{\alpha_s^{\overline{MS}}(\mu)}{\pi} + r_2^{\overline{MS}}(\frac{\alpha_s^{\overline{MS}}(\mu)}{\pi})^2 + \ldots \tag{186}$$

This is a general feature of perturbation theory. In this framework we necessarily deal with truncated series, so independence of the subtraction point or of the scheme can only be checked up to terms of the next order in the expansion. That is (R denotes some renormalization prescription)

$$\sum_{n=0}^{\infty} c_n(R)\alpha_s(R)^n = \sum_{n=0}^{\infty} c_n(R')\alpha_s(R')^n, \tag{187}$$

but

$$\sum_{n=0}^{N} c_n(R)\alpha_s(R)^n \neq \sum_{n=0}^{N} c_n(R')\alpha_s(R')^n; \tag{188}$$

the error being of $\mathcal{O}(\alpha_s^{N+1})$.

A lot of theoretical work has been done on the issue of which is the 'optimal' scheme. Some possibilities are

(i) FAC (Fastest Apparent Convergence): choose the scheme that makes $c_N = 0$, c_N being the last computed coefficient.

(ii) PMS (Principle of Minimal Sensitivity): demand that

$$\frac{\partial}{\partial R}(observable) = 0,$$

and work in the scheme R that fulfills this equation (which, of course, would be *any* scheme if we knew the observable exactly).

It is a fact that the quality of the series improves when one uses these methods, but unfortunately one is forced in general to use different schemes for different observables. On the basis of these analysis it has even been claimed that α_s somehow 'freezes' at ~ 0.3 at low energies. However, it is fair to say that the general properties of the perturbative series in QCD are so poorly understood that any method that does not directly rely on actually computing the neglected first term in the perturbative expansion and making sure that it is small is likely to be met with skepticism. It is hard to base derivations of fundamental parametres such as α_s on 'optimization' techniques.

9.1 R_τ

On energy considerations it is obvious that the τ is the only lepton we know heavy enough to decay into hadrons. This of course makes it a very interesting object. The inclusive decay rate $\tau \to \nu_\tau + hadrons$ can, in principle, be derived from QCD by exactly the same techniques as R_{had}. This is shown in fig. 12

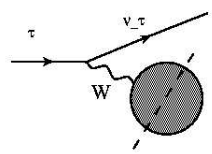

Figure 12: Determination of R_τ through dispersion relations.

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + hadrons)}{\Gamma(\tau \to \nu_\tau e \bar{\nu}_e)}. \tag{189}$$

At lowest order in QCD

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau d\bar{u}) + \Gamma(\tau \to \nu_\tau s\bar{u})}{\Gamma(\tau \to \nu_\tau e \bar{\nu}_e)} \simeq 3. \tag{190}$$

R_τ can be computed as a power series in α_s. Unlike in R_{had}, the typical scale will be very low (even zero) due to kinematical reasons (see fig. 12). To tackle this problem we decompose the W boson self-energy (which here plays the role the photon vacuum polarization did in R_{had}) into vector and axial parts as well as Cabibbo-allowed and Cabibbo-suppressed terms

$$\Pi^{\mu\nu} = |V_{ud}|^2(\Pi^{\mu\nu}_{udV} + \Pi^{\mu\nu}_{udA}) + |V_{us}|^2(\Pi^{\mu\nu}_{usV} + \Pi^{\mu\nu}_{usA}). \tag{191}$$

Each $\Pi^{\mu\nu}$ can be split into transverse and longitudinal parts

$$\Pi^{\mu\nu} = (-g^{\mu\nu}k^2 + k^\mu k^\nu)\Pi^{(1)} + k^\mu k^\nu \Pi^{(0)}. \tag{192}$$

For massless fermions $\Pi^{(0)} = 0$. We assume here that this is the case for simplicity. We can write

$$\begin{aligned} R_\tau &= 12\pi \int_0^{m_\tau^2} \tfrac{ds}{m_\tau^2}(1 - \tfrac{s}{m_\tau^2})^2(1 + \tfrac{2s}{m_\tau^2})\mathrm{Im}\Pi^{(1)}(s) \\ &= 6\pi i \int_{|s|=m_\tau^2} \tfrac{ds}{m_\tau^2}(1 - \tfrac{s}{m_\tau^2})^2(1 + \tfrac{2s}{m_\tau^2})\Pi^{(1)}(s) \end{aligned} \tag{193}$$

where we have used Cauchy theorem to write the integral we are interested in as a contour integral. For large $|s|$ we can compute $\Pi^{(1)}(s)$ just as we did for the hadronic vacuum polarization. One gets basically the same result (up to factors). For instance

$$\mathrm{Im}\Pi^{(1)}(s) = \frac{1}{2\pi}(|V_{ud}|^2 + |V_{us}|^2)[1 + \frac{\alpha_s(s)}{\pi} + r_2(\frac{\alpha_s(s)}{\pi})^2 + r_3(\frac{\alpha_s(s)}{\pi})^3 + \ldots]. \tag{194}$$

We also know that

$$\frac{\alpha_s(s)}{\pi} = \frac{\alpha_s(m_\tau^2)}{\pi} + \frac{\beta_1}{2}\frac{\alpha_s(m_\tau^2)}{\pi}\log\frac{s}{m_\tau^2} + \ldots \tag{195}$$

so finally,

$$R_\tau = 3(|V_{ud}|^2 + |V_{us}|^2)[1 + \frac{\alpha_s(m_\tau^2)}{\pi} + (r_2 - \frac{19}{24}\beta_1)(\frac{\alpha_s(m_\tau^2)}{\pi})^2 + \ldots + \text{n.p.t.}] \tag{196}$$

n.p.t. stands for non-perturbative contributions (contributions non-expressable as a power series in α_s). We did not have to worry too much about them in R_{had} because there they were characteristically suppressed by a power of the momentum transfer. Here they can be important. Fortunately, they are believed to be under control. Anyhow, we are now in posession of an alternative way of determining α_s though τ decay. The best data comes again from LEP. A fit to experimental numbers (including mass corrections, which have been neglected throughout) gives

$$\alpha_s(m_\tau) = 0.341 \pm 0.035 \qquad \Rightarrow \qquad \alpha_s(M_Z) = 0.121 \pm 0.004. \tag{197}$$

Thanks to the logarithmic scaling this method provides us with the most accurate determination of α_s so far. It is also a nice way of testing next to leading scaling.

9.2 Dispersion relations

Let us recall Cauchy's theorem

$$\Pi(q^2) = \frac{1}{2\pi i}\oint \frac{\Pi(s)}{s - q^2} ds \tag{198}$$

and the circuit of integration depicted in fig. 13

$\Pi(s)$ is assumed to be regular everywhere but on the positive real axis where it has a branch cut singularity plus, eventually, some poles. The circuit of integration thus avoid crossing that cut. The real part is continuous across the cut, while the imaginary part changes sign due to the logarithm and assuming further that the external integral can be neglected (i.e. that $\Pi(s)$ decays fast enough) we have

$$\Pi(q^2) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\Pi(s)}{s - q^2} ds \tag{199}$$

This analytic structure (certainly obeyed by QCD) is in fact one of the axioms of field theory: the two-point function is analytic everywhere except on the positive real axis where a number

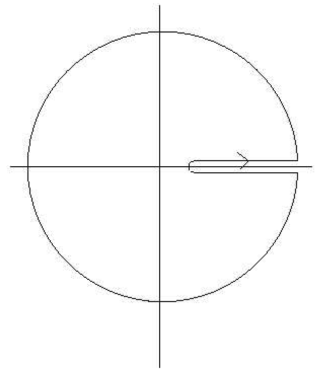

Figure 13: Integration circuit used to prove the dispersion relation.

of poles may exist. They correspond to delta-functions in the imaginary part of $\Pi(q^2)$. The log function in the complex plane is defined to have its cut along the negative real axis.

In perturbative QCD there are no poles and only the logarithmic cut that triggers the discontinuity in the imaginary part. However, Green functions do not typically have a good ultraviolet behaviour. Then, in order to neglect the exterior circuit, some derivatives have to be performed. This defines the dispersion relations up to an additive polinomial in the external momenta, just reflecting the arbitrariness in the renormalization procedure.
In QCD we can compute these Green functions explicitly, so the dispersion relation is just a tautology, but as we have seen in tau decay it may be actually quite useful. If a given Green function converges rapidly we can obtain it from the knowledge of its imaginary part alone.

It should be obvious that the imaginary parts come from the contribution of on-shell intermediate states. Recall

$$\frac{1}{p^2 - m^2 \pm i\epsilon} = P\frac{1}{p^2 - m^2} \mp i\pi\delta(p^2 - m^2) \tag{200}$$

Exercise.- Compute $\sigma(e^+e^- \to \mu^+\mu^-)$ directly from dispersion relations. The computation for $m_\mu = 0$ is much easier, but you are encouraged to try it for $m_\mu \neq 0$!