

Secular effects on inflation from one-loop quantum gravity

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Abstract

In this paper we revisit and extend a previous analysis where the possible relevance of quantum gravity effects in a cosmological setup was studied. The object of interest are non-local (logarithmic) terms generated in the effective action of gravity due to the exchange in loops of massless modes (such as photons or the gravitons themselves). We correct one mistake existing in the previous work and discuss the issue in a more general setting in different cosmological scenarios. We obtain the one-loop quantum-corrected evolution equations for the cosmological scale factor up to a given order in a derivative expansion in two particular cases: a matter dominated universe with vanishing cosmological constant, and in a de Sitter universe. We show that the quantum corrections, albeit tiny, may have a secular effect that eventually modifies the expansion rate. For a de Sitter universe they tend to slow down the rate of the expansion, while the effect seems to have the opposite sign in a matter dominated universe. To partly understand these effects we provide a complementary newtonian analysis.

1 Introduction

It has been said [1] that the effective action of quantum gravity is the most effective of all effective actions, meaning that an expansion in powers of $p^2/16\pi M_P^2$ would give in normal conditions such a tiny contribution to any scattering amplitude that the $\mathcal{O}(p^2)$ (the usual Einstein-Hilbert) term is good enough for all practical purposes (and even for many non-practical ones). This is unlike pion physics where the presence of higher order operators leads to measurable effects already at moderate energies. Thus, the fact that the $\mathcal{O}(p^4)$ terms are ultraviolet divergent does not really bother us in practical calculations¹, although of course the issue is very relevant from a fundamental point of view.

It is easy to see why quantum corrections are so small. In fact, as already implied above, the expansion is on powers of $p^2/16\pi M_P^2$ (actually $\nabla^2/16\pi M_P^2$, $\mathcal{R}/16\pi M_P^2$) and therefore very small for physical values of the energy or curvature. Non local pieces in the effective action ($\sim \ln \nabla^2$), due to the presence of strictly massless modes, somewhat increase the relevance of higher order terms, but locally they are still negligible.

There are two reasons why such apparently hopelessly small corrections might nevertheless be relevant in a cosmological setting. One reason is that curvature was much larger at early stages of the universe. For instance, in a de Sitter universe $\mathcal{R} \sim H^2$. In an inflationary scenario $H^2 = 8\pi G V_0/3$, V_0 being the scale of inflation that is limited by CMB measurements to be

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¹In pion physics, and in quantum gravity too, one can make sense of non-renormalizable interactions at the expense of introducing more subtraction constants, one per independent operator in the derivative expansion.

$V_0^{1/4} \leq 10^{16}$ GeV. The Hubble constant H could have been as large as $H \sim 10^{13}$ GeV (the present value is 10^{-42} GeV). However, even in the most favourable case, the correction is still nominally of $\mathcal{O}(10^{-12})$ or less and it seems too small to be seen. Or maybe not?

Indeed a second reason to study this problem is that this nominal suppression overlooks the fact that the logarithmic non local term corresponds to an interaction between geometries that is long-range in time, an effect that does not have an easy classical interpretation. When integrated over time, this may bring about a large enhancement of the contribution of higher order contributions to the point where new interesting effects may appear.

It is quite important to realize that there is no ambiguity in the overall coefficient of the logarithmic non-local term as it depends only on the structure of the Einstein-Hilbert Lagrangian and the number of massless modes (or modes whose mass is much smaller than the inverse of the horizon radius). Thus the effects are model independent and, if observable, can be unambiguously predicted, at least inasmuch as one can make accurate predictions in an effective theory.

The possibility of observable effects of the non-local terms in the effective action in such a situation was recognized in [2]. Although the results in [2] appear to be correct in substance, an error slipped in the calculation, unfortunately. This error is corrected here. More importantly the analysis is extended to different cosmological models. It is seen that a secular effect from the non-local terms can be predicted unambiguously (within the approximations inherent to an effective action, that is up to terms with higher derivatives) and it is seen to lead to potentially visible effects.

The relevance of quantum gravity corrections on inflation was also pointed out in [3]. Unfortunately, it is difficult to draw a parallelism between the two approaches. For one thing, we are finding here a one-loop effect, while the one discussed in [3] is a two-loop one due to particle creation [4] and thus clearly subleading.

The importance of non-local terms² in the effective action of gravity cannot be overemphasized. This has been recently reviewed in [5], although the non-localities discussed in that paper do not actually correspond to the one present here.

In section 2 we rederive the loop-corrected evolution equation for the cosmological scale factor in the presence of non-local logarithmic terms. In section 3 we apply these techniques to a matter dominated universe, governed by a power-law expansion. In section 4 we rederive the quantum corrections to the cosmological evolution equation in a de Sitter background. In section 5 we present the numerical analysis of the solutions and comment on their physical relevance. In order to better understand the relevance of the logarithmic terms we have provided a classical analogy that is presented in section 6.

The metric convention we use in this paper for Minkowski space is

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (1)$$

The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu}, \quad (2)$$

²By non-local we mean terms non-analytic in ∇^2 , such as the $\ln \nabla^2$ pieces that appear in one-loop effective gravity. We do not consider the so-called $f(\mathcal{R})$ gravity.

where $g_{\mu\nu}$ is the metric tensor, Λ is the cosmological constant, $R = g^{\mu\nu}R_{\mu\nu}$ and

$$R_{\mu\nu} = \partial_\nu \Gamma_{\mu\alpha}^\alpha - \partial_\alpha \Gamma_{\mu\nu}^\alpha + \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta - \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta \quad (3)$$

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\rho} (\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta}). \quad (4)$$

The previous equations are derived from the action

$$S = \frac{1}{16\pi G} \int dx \sqrt{-g} (\mathcal{R} - 2\Lambda) + S_{matter}. \quad (5)$$

Quantum corrections to the Einstein-Hilbert action were originally computed by 't Hooft and Veltman in [6] in the case of vanishing cosmological constant, and by Chistensen and Duff for a de Sitter background [7]. Other related references that we have found particularly useful in the present context are [8] and [9].

The key ingredient we shall need is the divergent part of the one-loop effective action. Using dimensional regularization and setting $d = 4 + 2\epsilon$ we get [9]

$$\Gamma_{eff}^{div} = -\frac{1}{16\pi^2\epsilon} \int dx \sqrt{-g} [c_1 \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + c_2 \Lambda^2 + c_3 \mathcal{R} \Lambda + c_4 \mathcal{R}^2]. \quad (6)$$

The constants c_i are actually gauge dependent and only a combination of them is gauge invariant. Using the equations of motion (in absence of matter) $\mathcal{R}_{\mu\nu} = g_{\mu\nu} \Lambda$, the previous equation reduces to the (gauge-invariant) on-shell expression

$$\Gamma_{eff}^{div} = \frac{1}{16\pi^2\epsilon} \int dx \sqrt{-g} \frac{29}{5} \Lambda^2. \quad (7)$$

If we set $\Lambda = 0$ in (6), we get the well-known 't Hooft and Veltman divergence, that in the so-called minimal gauge is

$$\Gamma_{eff}^{div} = -\frac{1}{16\pi^2\epsilon} \int dx \sqrt{-g} \left[\frac{7}{20} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{1}{120} \mathcal{R}^2 \right]. \quad (8)$$

If the equations of motion are used in the absence of matter this divergence is absent, as is well known.

Once the divergence is determined it is straightforward to obtain the logarithmic pieces since they always appear in the combination

$$\frac{1}{\epsilon} + \ln \frac{\nabla^2}{\mu^2}. \quad (9)$$

The dimensionful constant μ is the subtraction scale that is required for dimensional consistency.

2 The equations of motion in the presence of non-local terms

In this section we shall derive quantum corrected equations of motion for the cosmological scale factor including the non-local logarithmic terms that appear in the one-loop effective action.

We shall consider here a simplified effective action that includes only terms containing the scalar curvature we split the action into three parts and redefine the constants for convenience

$$\begin{aligned}
S &= \kappa^2 \left(\int dx \sqrt{-g} \mathcal{R} + \tilde{\alpha} \int dx \sqrt{-g} \mathcal{R} \ln(\nabla^2/\mu^2) \mathcal{R} + \tilde{\beta} \int dx \sqrt{-g} \mathcal{R}^2 \right) \\
&\equiv \kappa^2 \left(S_1 + \tilde{\alpha} S_2 + \tilde{\beta} S_3 \right),
\end{aligned} \tag{10}$$

where $\kappa^2 = M_P^2/16\pi = 1/16\pi G$. μ is the subtraction scale whose contribution is by itself local, but gives the right dimensions to the non-local term. The coupling $\tilde{\beta}$ is μ dependent in such a way that the total action S is μ -independent. While the value of $\tilde{\beta}$ is actually dependent on the UV structure of the theory (it contains information on all the modes -massive or not- that have been integrated out), the value of $\tilde{\alpha}$ is unambiguous as it depends only on the IR structure of gravity, that is entirely described by the Einstein-Hilbert Lagrangian and the massless modes.

In conformal time, $dt = a d\tau$, we have

$$g_{\mu\nu} = a^2(\tau) \eta_{\mu\nu}, \quad \mathcal{R} = 6 \frac{a''(\tau)}{a^3(\tau)}, \quad \sqrt{-g} = a^4(\tau). \tag{11}$$

In order to obtain the modified equations of motion for the cosmological scale factor, we first perform the variation of the local action, getting the following results

$$\frac{\delta S_1}{\delta a(\tau)} = 12a'' \tag{12}$$

$$\frac{\delta S_3}{\delta a(\tau)} = 72 \left(-3 \frac{(a'')^2}{a^3} - 4 \frac{a' a'''}{a^3} + 6 \frac{(a')^2 a''}{a^4} + \frac{a^{(4)}}{a^2} \right) \tag{13}$$

The d'Alembertian in conformal space is related to the Minkowski space operator by

$$\nabla^2 = a^{-3} \square a + \frac{1}{6} \mathcal{R} \tag{14}$$

Neglecting the \mathcal{R} term in the previous equation and commuting the scale factor a with the flat d'Alembertian (terms with higher derivatives are neglected in the effective action philosophy), we can write

$$\nabla^2 = \left(\frac{a}{a_0} \right)^{-2} \square \tag{15}$$

Where $a_0 = a(0)$. With this rescaling (absorbable in $\tilde{\beta}$), at $\tau = 0$ the d'Alembertian in conformal space matches with the Minkowskian one. We will set $a_0 = 1$ for simplicity.

We can now separate S_2 in turn into a local and a genuinely non-local piece

$$\begin{aligned}
S_2 &= \int dx \sqrt{-g} \left(-2\mathcal{R} \ln(a) \mathcal{R} + \mathcal{R} \ln(\square/\mu^2) \mathcal{R} \right) \\
&\equiv S_2^I + S_2^{II}.
\end{aligned} \tag{16}$$

The variation of S_2^I gives

$$\frac{\delta S_2^I}{\delta a(\tau)} = -72 \left\{ \frac{(a')^2 a''}{a^4} [12 \ln(a) - 10] + \frac{a' a'''}{a^3} [-8 \ln(a) + 4] + \right. \tag{17}$$

$$\left. + \frac{(a'')^2}{a^3} [-6 \ln(a) + 2] + \frac{a^{(4)}}{a^2} 2 \ln(a) \right\} \tag{18}$$

In order to determine the variation of S_2^{II} we need to compute the Green function

$$\langle x | \ln \square | y \rangle. \quad (19)$$

We follow the method of [2] that we shall not repeat here. We mention here that the normalization of the delta function used in [2] was non-covariant³. Using the proper normalization and correcting for this mistake we find

$$S_2^{II} = 36 \int d\tau \frac{a''(\tau)}{a(\tau)} \int_0^\tau d\tau' \frac{1}{\tau - \tau'} \frac{a''(\tau')}{a(\tau')} \quad (20)$$

The variation of S_2^{II} is

$$\begin{aligned} \frac{\delta S_2^{II}}{\delta a(\tau)} = 36 \left\{ \left[2a^{-3}(\tau) (a'(\tau))^2 - 2a^{-2}(\tau) a''(\tau) \right] \int_0^\tau d\tau' \frac{1}{\tau - \tau'} \frac{a''(\tau')}{a(\tau')} \right. \\ \left. - 2a^{-2}(\tau) a'(\tau) \frac{\partial}{\partial \tau} \left(\int_0^\tau d\tau' \frac{1}{\tau - \tau'} \frac{a''(\tau')}{a(\tau')} \right) + a^{-1}(\tau) \frac{\partial^2}{\partial \tau^2} \left(\int_0^\tau d\tau' \frac{1}{\tau - \tau'} \frac{a''(\tau')}{a(\tau')} \right) \right\} \quad (21) \end{aligned}$$

Using repeatedly integration by parts it is possible to further simplify the expression eliminating the derivatives acting on the integrals

$$\begin{aligned} \frac{\partial}{\partial \tau} \left(\int_0^\tau d\tau' \frac{1}{\tau - \tau'} \frac{a''(\tau')}{a(\tau')} \right) = \frac{a_0^{-1} a_0''}{\tau} + \\ + \int_0^\tau d\tau' \frac{1}{\tau - \tau'} \left[-a^{-2}(\tau') a'(\tau') a''(\tau') + a^{-1}(\tau') a'''(\tau') \right] \quad (22) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial \tau^2} \left(\int_0^\tau d\tau' \frac{1}{\tau - \tau'} \frac{a''(\tau')}{a(\tau')} \right) = -\frac{a_0^{-1} a_0''}{\tau^2} + \frac{-a_0^{-2} a_0' a_0'' + a_0^{-1} a_0'''}{\tau} + \\ + \int_0^\tau d\tau' \frac{1}{\tau - \tau'} \left[2a^{-3}(\tau') (a'(\tau'))^2 a''(\tau') - a^{-2}(\tau') (a''(\tau'))^2 \right. \\ \left. - 2a^{-2}(\tau') a'(\tau') a'''(\tau') + a^{-1}(\tau') a^{(4)}(\tau') \right] \quad (23) \end{aligned}$$

The terms that potentially diverge at $\tau = 0$ arise from the fact that we patch together Minkowski and FRW spaces at $\tau = 0$. If it is done smoothly enough, the derivatives of the scale factor should vanish at that point.

The previous expressions are formal in the sense that the short distance singularities have not been properly regularized. To regulate the region where $\tau \rightarrow \tau'$ we use dimensional regularization as explained in detail in [2]. The regulator is in fact introduced when the Green function (19) is computed in $4 + 2\epsilon$ dimensions. In practice, this amounts to the replacement

$$\frac{1}{\tau - \tau'} \rightarrow \frac{\mu^{-2\epsilon}}{(\tau - \tau')^{1+2\epsilon}}, \quad (24)$$

where μ is the subtraction scale previously introduced. The physical result is obtained for $\epsilon \rightarrow 0$. The logarithm of the effective action (10) is then reproduced and the $1/\epsilon$ divergence (proportional to \mathcal{R}^2) is cancelled by the (divergent) counterterm included in $\tilde{\beta}$. Thus it is easy

³We thank G.Pérez for bringing this to our attention

to compute, even numerically, the genuinely non-local piece, as the would-be singular term in the integral is clearly identified.

At this point there are several ways to proceed. One might of course attempt to find solutions of the equation of motion for $a(\tau)$ obtained by adding the variations for S_1 , $\tilde{\beta}S_3$ and $\tilde{\alpha}(S_2^I + S_2^{II})$ that we have just computed. This way of proceeding is not really justified if the $\mathcal{O}(p^4)$ terms are understood as a correction.

In the spirit of effective Lagrangians it is better to proceed otherwise. We obtain first the lowest order equation of motion from S_1 and plug it in $\tilde{\alpha}(S_2^I + S_2^{II}) + \tilde{\beta}S_3$. The quantum corrections act then as an external driving force superimposed to Einstein equations. This procedure of course gives trivially a net zero additional contribution in the present toy model as neither matter nor a cosmological constant have been considered. In the next sections we shall introduce $T_{\mu\nu}$ and Λ to find more interesting effects.

3 Quantum gravity effects in a matter dominated universe

Let us consider a pressureless distribution of matter characterized by the energy-momentum tensor $T_{\mu\nu} = \text{diag}(\rho, 0, 0, 0)$. We can use Einstein equations to write (in conformal time)

$$\mathcal{R}_{00} = -4\pi G a^2(\tau)\rho, \quad \mathcal{R}_{ij} = 4\pi G a^2(\tau)\rho. \quad (25)$$

The addition of an energy-momentum tensor leads in principle to the appearance of $\mathcal{R}T$, T^2 and $T_{\mu\nu}T^{\mu\nu}$ terms in the effective action. However, the details of the calculation are still controversial to some extent. Here we shall adopt the view that a classical energy-momentum tensor (such as one representing pressureless dust) interacts only via classical gravitons.

We then substitute these expressions in the non-local $\mathcal{O}(p^4)$ term effective action

$$S = \frac{1}{16\pi G} \int dx \sqrt{-g} \mathcal{R} \quad (26)$$

$$- \frac{1}{16\pi^2} \int dx \sqrt{-g} \left[\frac{7}{20} \mathcal{R}^\mu{}_\nu \ln \frac{\nabla^2}{\mu^2} \mathcal{R}^\nu{}_\mu + \frac{1}{120} \mathcal{R} \ln \frac{\nabla^2}{\mu^2} \mathcal{R} \right] \quad (27)$$

$$+ \text{local terms of } \mathcal{O}(p^4) + S_{matter}. \quad (28)$$

The local $\mathcal{O}(p^4)$ terms (proportional to \mathcal{R}^2 and $\mathcal{R}^\mu{}_\nu \mathcal{R}^\nu{}_\mu$) can be used to absorb the divergences of the one-loop action. If one day one would be able to make a precise measurement of some quantum gravity effect, one could determine these coefficients and all other predictions would be calculable unambiguously with an $\mathcal{O}(p^4)$ precision. For the time being, we only know unambiguously the logarithmic non-local terms. But these are the ones that are like to give enhanced contributions as we discussed in the introduction.

After use of the lowest order, $\mathcal{O}(p^2)$ equations of motion, the non local terms simplify considerably

$$S = \kappa^2 \left(\int dx \sqrt{-g} \mathcal{R} + \tilde{\alpha} \int dx \sqrt{-g} \rho \ln \frac{\nabla^2}{\mu^2} \rho \right) + S_{matter} \quad (29)$$

$$\equiv \kappa^2 \left(\int dx \sqrt{-g} \mathcal{R} + \tilde{\alpha} S_2 \right) + S_{matter}. \quad (30)$$

We omit the purely local part (analogous to S_3 in the previous section) as is not calculable from the low energy information only and logs dominate anyway.

The value of $\tilde{\alpha}$ can be determined readily from the results of 't Hooft and Veltmann [6] (which includes only the contribution of virtual gravitons) and the lowest order equations of motion in the presence of matter

$$\tilde{\alpha} = -16\pi G^3 \times \frac{43}{30}. \quad (31)$$

The correction from massless photons or Yang-Mills fields does not seem to change the sign of $\tilde{\alpha}$ [10]. We have not considered other possibilities. In fact, the precise value of $\tilde{\alpha}$ is not so important (but the sign and its rough order of magnitude is).

S_2 modifies the equations of motion by adding a new term that is simply

$$\frac{\delta S_2}{\delta a(\tau)} = -2\rho(t)^2 a^3(\tau)[4\ln(a(\tau)) + 1] + 2\rho(\tau)a(\tau) \int_0^\tau \rho(\tau')a^2(\tau') \frac{\mu^{-2\epsilon}}{|\tau - \tau'|^{1+2\epsilon}}, \quad (32)$$

multiplied by $\tilde{\alpha}$. Notice that quantum effects introduce long range interactions in time between global matter densities at different times.

The value of $\rho(\tau)$ is known from the lowest order equation of motion. In conformal time

$$\rho(\tau) \sim a^{-3}(\tau), \quad a(\tau) \sim \tau^2. \quad (33)$$

We shall discuss the physical relevance of these corrections after discussing in detail the solution in the de Sitter case.

4 Quantum gravity effects in a de Sitter universe

We shall proceed in a way similar to the previous section, but we shall now omit the energy-momentum tensor in the lowest order equations of motion. In fact it is not correct to simply assume $T_{\mu\nu} \sim g_{\mu\nu}\Lambda$ in the equations of motion and just use the previous formulae (that would roughly be equivalent to exchanging ρ by Λ in the previous section, up to constants). The cosmological constant is the most relevant operator in gravity and it must be introduced from the outset.

The relevant one-loop corrected effective action is

$$S = \frac{1}{16\pi G} \int dx \sqrt{-g}(\mathcal{R} - 2\Lambda) + \frac{1}{16\pi^2} \int dx \sqrt{-g} \frac{29}{5} \Lambda \ln \frac{\nabla^2}{\mu^2} \Lambda + \text{local terms of } \mathcal{O}(p^4) \quad (34)$$

$$\equiv \kappa^2 \left(\int dx \sqrt{-g}(\mathcal{R} - 2\Lambda) + \tilde{\alpha} S_2 \right). \quad (35)$$

Now $\tilde{\alpha}$ is very different from the previous case

$$\tilde{\alpha} = \frac{G}{\pi} \times \frac{29}{5} \quad (36)$$

The dimensions of $\tilde{\alpha}$ are of course different as the dimensionality of ρ and Λ is not the same. Most importantly, it has the opposite sign. Notice that all our expressions are written in such a way that it is possible to consider a time-dependent cosmological constant (or matter density).

We split S_2 in two parts

$$S_2^I = -2 \int dx \sqrt{-g} \Lambda^2 \ln(a) \quad (37)$$

$$S_2^{II} = \int dx \sqrt{-g} \Lambda \ln(\square/\mu^2) \Lambda, \quad (38)$$

and obtain the corresponding variations following the method outlined in section 2

$$\frac{\delta S_2^I}{\delta a(\tau)} = -2\Lambda^2 a^3(\tau) [4 \ln(a(\tau)) + 1] \quad (39)$$

$$\frac{\delta S_2^{II}}{\delta a(\tau)} = 2\Lambda^2 a(\tau) \int_0^\tau d\tau' a^2(\tau') \frac{\mu^{-2\epsilon}}{|\tau - \tau'|^{1+2\epsilon}}. \quad (40)$$

The equation of motion will be

$$12a''(\tau) - 8\Lambda a^3(\tau) + \tilde{\alpha} \frac{\delta S_2}{\delta a(\tau)} = 0, \quad (41)$$

which at lowest order is just

$$12a''(\tau) - 24H^2 a^3(\tau) = 0, \quad (42)$$

where $H^2 = \Lambda/3$. The solution of (42) is

$$a_I(\tau) = \frac{1}{1 - H\tau}. \quad (43)$$

The final step to solve iteratively (41) is to plug the 0-th order solution $a_I(\tau)$ into the variation of S_2 and recalculate the solution for $a(\tau)$.

It is clear that, apart from the sign difference, the quantum effects are formally very similar for a matter dominated and for a de Sitter universe. However, the fact that the signs are opposite means that their back reaction is completely opposite. If quantum corrections enhance expansion in one case, they will slow it down in the other. Furthermore, the size of the correctios is very different: the corrections in a matter dominated universe are down by a factor H^2/M_P^2 with respect to the ones in a de Sitter space-time with a large cosmological constant, already expected to be small. Let us now investigate the numerical relevance of the latter ones.

5 Solving the evolution equation

As we just discussed, we proceed by solving the varied gravitational action by a perturbative approximation, i.e., we consider the non-standard terms as a correction to the standard inflationary solution. This perturbative procedure is of course only valid as long as the correction is small compared to the unperturbed solutions.

Before doing that we find it convenient to change time coordinates by introducing a variable s defined through $a_I = e^s$. Then

$$\left. \frac{\delta S_2^I}{\delta a(\tau)} \right|_{a_I} = -2\Lambda^2 e^{3s} [4s + 1] \quad (44)$$

$$\left. \frac{\delta S_2^{II}}{\delta a(\tau)} \right|_{a_I} = 2\Lambda^2 e^s I(s) \quad (45)$$

and the equation of motion reads

$$e^{2s} a''(s) + e^{2s} a'(s) - 2a^3(s) = \frac{3}{2} \tilde{\alpha} H^2 (-e^{3s}(1+4s) + e^s I(s)), \quad (46)$$

where I is defined in conformal time as

$$I(\tau) \equiv \mu^{-2\epsilon} \int_0^\tau d\tau' \frac{a_I^2(\tau')}{(\tau - \tau')^{1+2\epsilon}} = -\frac{1}{2\epsilon} (\tau\mu)^{-2\epsilon} {}_2F_1(1, 2; 1 - 2\epsilon; H\tau) \quad (47)$$

with ${}_2F_1$ being a hypergeometric function. Let us expand this expression around $\epsilon = 0$ to the first order, disregarding higher orders since we eventually take the limit $\epsilon \rightarrow 0$. Using ${}_2F_1(1, n; 1; H\tau) = (1 - H\tau)^{-n} = a_I^n(\tau)$, we get

$$\begin{aligned} I(\tau) &= \ln(\tau\mu) {}_2F_1(1, 2; 1; H\tau) - \frac{1}{2} \frac{\partial}{\partial \epsilon} \left[{}_2F_1(1, 2; 1 - 2\epsilon; H\tau) \right] \Big|_{\epsilon=0} = \\ &= \ln(\tau\mu) a_I^2(\tau) - \frac{1}{2} \frac{\partial}{\partial \epsilon} \left[{}_2F_1(1, 2; 1 - 2\epsilon; H\tau) \right] \Big|_{\epsilon=0}. \end{aligned} \quad (48)$$

This can be computed and written in s time as

$$I(s) = \ln\left(\frac{\mu}{H}(1 - e^{-s})\right) e^{2s} + e^s(1 - e^s - se^s), \quad (49)$$

and the equation to solve is

$$e^{2s} a''(s) + e^{2s} a'(s) - 2a^3(s) = \frac{3}{2} \tilde{\alpha} H^2 \left[-(5s+2)e^{3s} + e^{2s} + e^{3s} \ln\left(\frac{\mu}{H}(1 - e^{-s})\right) \right] \quad (50)$$

Note that $\tilde{\alpha}$ appears only in the combination $\tilde{\alpha} H^2$. Since there are H large uncertainties in H in practice only the sign of $\tilde{\alpha}$ is relevant. In addition, there is some ambiguity associated to the choice of the renormalization scale that appears in the combination $\ln(\mu/H)$ ⁴. The dependence on the subtraction scale is mild (logarithmic) but it is inherent to the effective action approach. To reverse the sign of the effect one has to go to absolutely unreasonable values of μ .

Eq. (50) can now be solved for different values of H and μ , where $H \lesssim 10^{13}$ GeV. The solution is shown in Fig. 1. We can see that the curves present a very similar behaviour for the different values shown, though a higher value of H leads earlier to deviations from the usual inflationary expansion. Higher values of μ also have this effect, which is larger as H increases (note that what is relevant in the equation is the ratio μ/H). In fact, we can see from (50) that if we considered values of μ/H large enough (but not relevant physically), the logarithm term would become dominant and the deviation would be positive.

6 Newtonian approximation

It is well known that one can derive the Friedmann equations using only newtonian physics [11]. We will redo this exercise considering quantum corrections to the Newton potential. These have been computed by several authors [12]. In fact, the magnitude of this correction

⁴Recall that a change of μ is equivalent to a redefinition of the local $\mathcal{O}(p^4)$ counterterms

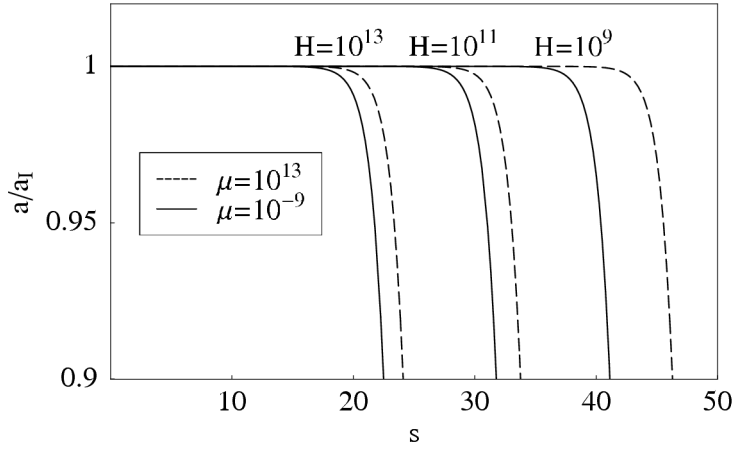


Figure 1: The scale factor relative to the inflationary expansion for different values of the renormalization scale μ and the Hubble constant H (all units are GeV). s gives roughly the number of e-folds. The precise definition is given in the text.

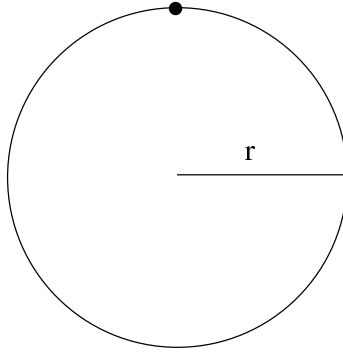


Figure 2: In order to derive the Friedmann equation, we consider a test particle on the surface of a virtual sphere in our infinite, homogeneous and isotropic universe.

and even the sign has been the subject of a long controversy. The correct value for the quantum correction to the gravitational force between two masses is given for instance in [13]. This we believe to be the relevant modification in the context of this newtonian approximation, but we shall redo the analysis for an arbitrary value of the correction for the sake of completeness.

Quantum corrections can be summarized in a modification of Newton's 'constant' in the following way

$$G(r) = G \left(1 + \xi \frac{G\hbar}{r^2} \right), \quad (51)$$

where ξ is a dimensionless constant.

We consider an infinite, homogeneous and isotropic expanding universe. We will describe the equations of movement for a test particle on the surface of a comoving sphere, expanding with the universe. The mass inside the sphere is $\rho(4\pi/3)r^3$, where ρ is the density of the universe and r the radius of the sphere. (see Fig. 2). The kinetic energy of the (unit mass) particle is $(1/2)\dot{r}^2$. In order to compute its potential energy U we shall consider first the force

between our test particle and another particle with mass m at a distance l from it

$$\vec{F} = -G(r)m\frac{\hat{l}}{l^2} = -Gm\frac{\hat{l}}{l^2} - \xi G^2 \hbar m \frac{\hat{l}}{l^4} \equiv \vec{F}_1 + \vec{F}_2 \quad (52)$$

The first part \vec{F}_1 is of course elementary and once integrated it leads to a contribution to the total potential energy $U = U_1 + U_2$ that is easy to guess

$$U_1 = -\frac{GM}{r} = -\frac{4\pi G\rho}{3}r^2 \quad (53)$$

For the second part, we have to integrate \vec{F}_2 over the whole sphere and also the space outside of it, since we can not make use of Gauss theorem. After doing this we get the following result for the contributions from matter outside and inside the sphere, respectively

$$\begin{aligned} \vec{F}_{2_{out}} &= \frac{4\pi}{3} (\xi G^2 \hbar) \rho \left(\frac{r}{r^2 - \tilde{r}^2} \Big|_{\tilde{r} \rightarrow r^+} \right) \hat{r} \\ \vec{F}_{2_{in}} &= -\frac{4\pi}{3} (\xi G^2 \hbar) \rho \left(\frac{1}{r} + \frac{\tilde{r}}{\tilde{r}^2 - r^2} \Big|_{\tilde{r} \rightarrow r^-} \right) \hat{r} \end{aligned} \quad (54)$$

Where \hat{r} is the unit radial vector pointing to increasing values of r . If we sum these two contributions, the divergences in the boundary cancel and we have

$$\begin{aligned} \vec{F}_2 &= -(\xi G^2 \hbar) \frac{M}{r^4} \hat{r} \\ U_2 &= -(\xi G^2 \hbar) \frac{M}{3r^3} = -\frac{4\pi}{3} (\xi G^2 \hbar) \frac{\rho}{3} \end{aligned} \quad (55)$$

This is a manifestation of Birkhoff's theorem.

We can now write the total energy of our test particle, that is

$$E = \frac{1}{2}\dot{r}^2 - \frac{4\pi G\rho}{3}r^2 - \frac{4\pi}{3} (\xi G^2 \hbar) \frac{\rho}{3} \quad (56)$$

Writing $r(t) = a(t)x$ and dividing by $a^2/2$ we get the modified Friedmann equation.

$$\left(\frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi G}{3} \rho + \frac{8\pi}{3} (\xi G^2 \hbar) \frac{\rho}{3} \frac{1}{a^2(t)x^2} - \frac{K}{a^2(t)} \quad (57)$$

Where $K = -2E/x^2$ is the curvature, which we set to 0 since we are considering a flat space. As we see, the local coordinate x appears explicitly and in order to put precise numbers we have to be very precise as we define our unit system. This is not very relevant for practical purposes due to the smallness of the correction and we shall embed this into ξ .

After identifying the cosmological constant $\Lambda = 8\pi G\rho$, we have

$$\left(\frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{\Lambda}{3} \left(1 + \frac{\xi G \hbar}{3} \frac{1}{a^2(t)} \right). \quad (58)$$

In order to compare with the time coordinate we have introduced before, we revert to the variable s , remembering that, to the lowest order, $H^2 = \Lambda/3$.

$$(a'(s))^2 e^{2s} = a^4(s) \left(1 + \frac{\xi G \hbar}{3} \frac{1}{a^2(s)} \right) \quad (59)$$

The resulting evolution can be computed easily and the effects are seen to be very small; so small in fact that to make them visible in a reasonable time evolution (comparable to the times seen in Fig. 1) we have taken $\xi G\hbar \sim 10^{-20}$, which is an unreasonable value for sure (see Fig. 3). The effect is to increase the expansion rate. This simple exercise teaches us two things. One is that it is incorrect to simply take the cosmological constant as part of the ‘matter balance’ of the universe and stick to newtonian physics. Incidentally, the calculation would be the same for a matter dominated universe and the effects even tinier. The second thing we learn is that classical physics with a potential modified by quantum corrections is unable to reproduce the enhancement that the non-local terms bring about due to long range time correlations.

7 Conclusions

In this paper we have analyzed the relevance of the non-local quantum corrections due to the virtual exchange of gravitons and other massless modes to the evolution of the cosmological scale factor in FRW universes. We have considered two different setups: a matter dominated universe, characterized by a matter density $\rho(t)$, and a de Sitter universe with a large cosmological constant.

In the de Sitter universe, while the effects are locally absolutely tiny, even after allowing for the largest possible value of H , we have found that they lead to a noticeable secular effect that slows down the inflationary expansion after a long time. This is a pure one-loop effect that is not actually related to particle creation and its back-reaction on the universe expansion and which constitutes a two loop effect as emphasized by Tsamis and Woodard. In a matter dominated universe the effect is a lot smaller, and it may be of the opposite sign. It is quite interesting that quantum effects seem to enhance the expansion rate in this case.

The physical effects thus depend crucially on the sign of the quantum corrections. We have also seen that this effect has no really classical analogy.

It is quite important to emphasize once more that the results presented here are not ‘just another model’. Quantum gravity non-local loop corrections are required by unitarity, even if the theory is non-renormalizable; they can be computed quite precisely in a derivative expansion; and they appear to be of some relevance in the present situation. Perhaps the most interesting result of our analysis is indeed the fact that these effects can be predicted unambiguously within the limits of an effective theory.

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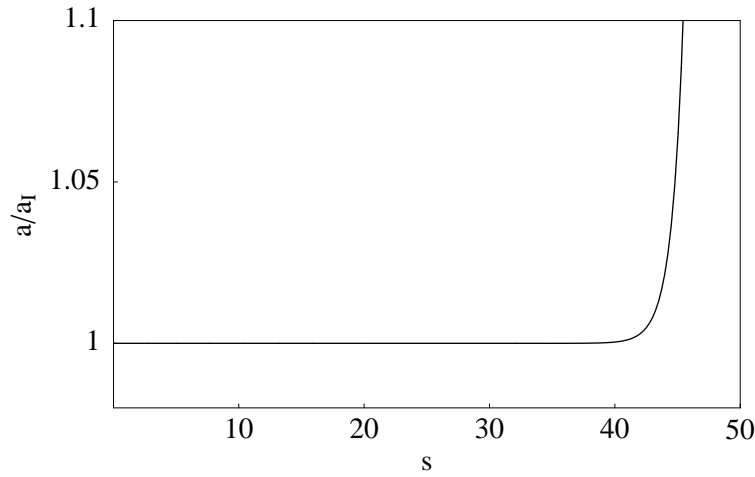


Figure 3: The scale factor relative to the inflationary expansion resulting from our Newtonian approach. This curve corresponds to $\xi G\hbar \sim 10^{-20}$.

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