

HADRONIC STRING, CONFORMAL INVARIANCE AND CHIRAL SYMMETRY

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While it is clear that in some kinematic regime QCD can be described by an effective (as opposed to fundamental) string theory, it is not at all clear how this string theory should be. The ‘natural’ candidate, the bosonic string, leads to amplitudes with the usual problems related to the existence of the tachyon, absence of the adequate Adler zero, and massless vector particles, not to mention the conformal anomaly. The supersymmetric version does not really solve most of these problems. For a long time it has been believed that the solution of at least some of these difficulties is associated to a proper identification of the vacuum, but this program has remained elusive. We show in this work how the first three problems can be avoided, by using a sigma model approach where excitations above the correct (chirally non-invariant) QCD vacuum are identified. At the leading order in a derivative expansion we recover the non-linear sigma model of pion interactions. At the next-to-leading order the $O(p^4)$ Lagrangian of Gasser and Leutwyler is obtained, with values for the coefficients that match the observed values. We also discuss some issues related to the conformal anomaly.

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1. Introduction

It seems that a paper with this title should necessarily start with a list of the reasons that make us believe that there should be an effective description of QCD in terms of strings¹.

The more commonly cited arguments are the dominance of planar diagrams in the large N limit² ‘filling in’ a surface (interpreted as the world-sheet of a string), the expansion in terms of surfaces built out of plaquettes in strong-coupling lattice QCD³, and the success of Regge phenomenology⁴, which can ultimately be understood in terms of string theory ideas (although, as we will discuss a little bit later, the actual Regge theory that corresponds to QCD cannot be derived, at present, from any known string theory).

To these we could add two more reasons. One is the appearance in string theory of the universal (at long distances) Lüscher term⁵. The static inter-quark potential provided by the string $V(r) = \sigma r + c$ gets modified by quantum fluctuations by a Coulomb-like piece $-\pi/12r$. While for the heavy quark bound state calculations it is perturbative QCD (complemented with some non-perturbative corrections) that is applicable, and not the string regime, the Lüscher term is quite useful. Finally, and in a completely different context, namely that of deep inelastic scattering, the evolution of the parton distribution down to low values of Q^2 (around $(2 \text{ GeV})^2$) leads⁶ to a low x behavior for the structure functions of the form $x^{-1.17}$, while Regge theory predicts x^{-1} , in striking good agreement.

In the last years the possibility of describing some non-abelian gauge theories in terms of string theories has become a reality thanks to the AdS/CFT relationship⁷. Unfortunately, the gauge theories that one can describe, or rather solve, in this way possess an unrealistically large number of supersymmetries. Breaking this large amount of supersymmetry to the realistic case of $N = 0$ while decoupling the unwanted degrees of freedom from long-distance physics has proven to be a formidable task and one in which not much success has been attained yet.

It should be clear to the reader from the very beginning that we are not addressing in this paper these deep and fundamental issues. Yet, it is clear that the mentioned theoretical developments give credence to the idea that there must be some kinematic regime where the simplest bosonic string, obtained after integrating out all the remaining degrees of freedom made heavy by the breaking of supersymmetry, must provide an approximately valid description.

In fact it is highly questionable where one should expect a string description to be exactly valid for real, non-supersymmetric, four-dimensional, asymptotically free QCD, as a string does not seem to be the natural language to understand high-energy processes in deep-inelastic scattering where the point-like structure of quarks and gluons is apparent. We should probably be less ambitious and satisfy ourselves with an effective description. We subscribe this point of view and think of strings as effective theories and not worry at all about their mathematical consistency as fundamental objects.

It is surprising that even with this limited scope all known string theories fail to provide a description of low energy QCD. To see how this comes about let us recall the original Veneziano amplitude⁸. The original motivation for this *ad hoc* formula was to provide a description of hadrons that manifestly exhibited duality. While the evidence for duality was rather weak⁹, at the time this amplitude sparked Nambu¹⁰ to suggest that it should originate in bosonic string theory.

After decorating the Veneziano amplitude with the appropriate Chan-Paton¹¹ factors^a, it is supposed to describe the scattering amplitude of four pions

$$\mathcal{A}(\pi^a \pi^b \rightarrow \pi^c \pi^d) \sim \text{Tr}(T^a T^b T^c T^d) A(s, t) + \text{non cyclic permutations}, \quad (1)$$

$$A(s, t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}, \quad (2)$$

where $\alpha(s) = 1 + \alpha' s$ is the Regge trajectory. The Regge trajectory slope is determined, after a fit, to be $\alpha' \simeq 0.9 \text{ GeV}^{-2}$ ¹² We immediately recognize that there are poles in the s -channel whenever $\alpha' s = n - 1$. Thus a tachyon is present for $n = 0$ and we see the first of the problems alluded to in the abstract appearing. Furthermore the second ($J = 1$) state (which we should identify with the rho particle) is massless. This is the second one of the problems we alluded to.

Shortly afterward, supersymmetry was introduced in string theory¹³, but unfortunately most of the difficulties with the bosonic string are still present in supersymmetric string theory when one applies it to the present context. The relevant amplitude for our purposes is the four tachyon amplitude^b

$$A(s, t) = \frac{\Gamma(1 - \alpha_\rho(s))\Gamma(1 - \alpha_\rho(t))}{\Gamma(1 - \alpha_\rho(s) - \alpha_\rho(t))}. \quad (3)$$

Now $\alpha_\rho(s) = 1 + \alpha' s$ is the so-called ρ -trajectory which in Regge theory describes exchanges of particles with positive G -parity. This is the Lovelace-Shapiro amplitude and it contains no tachyonic poles. Could it then be a candidate to describe pion scattering? The answer is no. It does not have the appropriate Adler zero, i.e. the property that at $s = t = 0$ the pion scattering amplitude vanishes. And although it has no tachyons in the intermediate channel, unfortunately, the vertex which is supposed to represent the pion is a tachyon itself. Furthermore, the first resonance in the ρ -trajectory (supposed to represent the physical ρ particle) has zero mass. All in all, one is at the same dead end.

A fix¹⁵ to this problem consists in arbitrarily changing the intercept in $\alpha_\rho(s)$. Indeed, if we write $\alpha_\rho(s) = \frac{1}{2} + \alpha' s$ and use Eq. (3), the resulting amplitude has poles in the s or t -channel when $\alpha' s = n + \frac{1}{2}$. It has no tachyons and the first

^aDue to difficulties with unitarity only orthogonal groups or certain representations of unitary groups can be introduced in this way.

^bIn supersymmetric theories one usually performs the GSO projection¹⁴, projecting out the tachyon. However one may choose not to do so and compute the four-tachyon amplitude, supposed to describe scattering of spinless particles in the bosonic (Neveu-Schwarz) sector.

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pole, which is massive, can be identified with the ρ particle. Furthermore, the previous amplitude has the right Adler zero. Expanding this amplitude in powers of s and t and comparing with the amplitudes obtained from an effective Lagrangian, Polyakov and Vereshagin¹⁶ found that $L_1 = \frac{1}{2}L_2$, $L_2 = \frac{F_\pi^2}{8m_\rho^2} \ln 2$, $L_3 = -2L_2$, which numerically turn out to be quite acceptable values^c. Unfortunately, no known string theory leads to such a fix in the intercept^d.

It is believed that the ultimate reason for the presence of a tachyon in the spectrum lies in a wrong choice of the vacuum²². Since the choice of the spin zero vertex operator, $V(k) =: \exp ikx :$, is based on the Lorentz properties alone, it is the same both for scalar and pseudoscalars and, accordingly, both scalars and pseudoscalars have tachyonic poles in the s -channel on account of parity conservation. The situation is thus parallel to the one in multicomponent ϕ^4 theory with $V(\phi) = -\mu^2 + \lambda\phi^4$, where perturbing around $\phi = 0$ gives negative m^2 values for all components. It is natural to assume that the amplitudes obtained through the use of the canonical vertex operators correspond to (unphysical) amplitudes for excitations perturbed around the wrong, unphysical vacuum. These ideas have in fact been around for a long time, but no one appears to have implemented them in any practical way.

In a previous paper²³ by some of the present authors a possible line of attack was suggested. The idea is the following. Given that with the present string theory technology it is impossible to find the ‘true’ vacuum, let us assume that the vacuum is non-perturbative in string creation and annihilation operators and that it actually corresponds to the true QCD vacuum. In this case, the relevant (lightest) degrees of freedom are the ones emerging after the spontaneous breaking of chiral symmetry. In the physical vacuum of QCD there is a clear distinction between scalars (sigma particle) and pseudoscalars (pions). The sigma particle is just another hadronic resonance with a mass in the 1 GeV region and without any specially important role to play. The massless pseudoscalars, on the other hand, can be collected in a unitary matrix $U(x)$ which under chiral transformations belonging to $SU(3)_L \times SU(3)_R$ transforms as $U(x) \rightarrow U'(x) = LU(x)R^\dagger$ and describes excitations around the non-perturbative vacuum. From the string point of view $U(x)$ is nothing but a bunch of couplings involving the string variable x . Our goal is to find a consistent string propagation in this non-perturbative background.

A crucial property of string theory is, undoubtedly, conformal invariance. This amounts to demanding that the theory is independent of the specific conformal factor chosen to describe the two-dimensional world sheet. While this is a desirable

^cThe relation $L_1 = 1/2L_2 = -1/4L_3$ was established earlier in bosonization models¹⁷ and in the chiral quark model¹⁸ by means of a derivative expansion of quark determinant. However at that time its possible connection with a string description of QCD was not recognized.

^dIt appears possible to modify to some extent the Regge trajectory for low values of J, M^2 , and hence the intercept by adding additional terms in the string action, rigidity, for instance^{19,20} or imposing constraints on the string motions²¹. However spectroscopy suggest that while such terms might exist they are probably of limited influence. Moreover the conformal invariance of a modified string becomes questionable.

and, as a matter of fact, a crucial property of fundamental strings, it might not be necessarily so for the QCD string (if we look at the QCD string with a magnifying glass we shall eventually see quarks and gluons, not the string itself!) . However in the limit of large N_c the QCD hadronic amplitudes are saturated by gluon fishnet diagrams among which any higher order samples give, in principle, a comparable contribution. Thereby it appears plausible that there should be no dependence on a specific choice of configuration describing a dominating gluon propagation. Therefore the hadronic string action should obey reparameterization invariance of diagram surfaces and conformal invariance. Since conformal invariance must hold when perturbing the string around any vacuum, perturbative or not, we have a powerful tool at our disposal, namely to couple the chiral field $U(x)$ to the string degrees of freedom and demand conformal invariance. Exactly in the same way as Einstein equations are derived from string theory²⁴ by requiring the vanishing of the beta function for an external metric $G_{\mu\nu}(x)$.

This approach was followed in ²³ with mixed success. The lowest order chiral Lagrangian, or rather the equations of motion coming from it, were obtained and the whole approach seemed consistent. Unfortunately, the $O(p^4)$ coefficients were identically zero. Experimentally these are numbers of order 10^{-3} . We have now found out the reason for this failure. It turns out that the action considered in ²³ is only a particular case of the general case, and one that does not allow a proper treatment of the unitarity condition of the external source $U(x)$. When the general case is considered, unitarity and locality can be implemented consistently—at least at the order we have worked—and, as a consequence of the new terms in the action, new divergences and hence new contributions to the beta function for $U(x)$ appear. These lead to $O(p^4)$ terms in the effective Lagrangian. These are the results we would like to report here.

2. Coupling pions to the QCD string

The hadronic string in the conformal gauge is described by the conformal field theory action in the four dimensional Euclidean space-time ^e

$$\mathcal{W}_{str} = \frac{1}{4\pi\alpha'} \int d^{2+\epsilon}\sigma \left(\frac{\varphi}{\mu} \right)^{-\epsilon} \partial_i x_\mu \partial_i x_\mu, \quad (4)$$

where for $\epsilon = 0$ one takes

$$x_\mu = x_\mu(\tau, \sigma); \quad -\infty < \tau < \infty, 0 < \sigma < \infty; \quad i = \tau, \sigma \quad \mu = 1, \dots, 4.$$

The conformal factor $\varphi(\tau, \sigma)$ is introduced to restore the conformal invariance in $2+\epsilon$ dimensions, and this is the only way it enters the theory. The Regge trajectory

^eThe direct evidences for dimensions of the string world sheet^{25,26} ($d=2$) and of the target space²⁷ ($D=4$) were found from the analysis of meson state densities at high energies as compared to the Hagedorn-like experimental growth of the latter ones.

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slope (related to the inverse string tension) is known to be universal $\alpha' \simeq 0.9 \text{ GeV}^{-2}$ ¹².

We would like to couple in a chiral invariant manner the matrix in flavor space $U(x)$ containing the meson fields to the string degrees of freedom while preserving general covariance in the two dimensional coordinates and conformal invariance under local scale transformations of the two-dimensional metric tensor. The equations of motion for the $U(x)$ field will then be obtained from the condition that the quantum theory must be conformally invariant, i.e. the β functional for the $U(x)$ couplings must vanish.

Since the string variable x does not contain any flavor dependence, we have to invent a way to couple it to the background $U(x)$ variable. We introduce two dimensionless Grassmann variables ('quarks'), or rather several families of them, living on the boundary of the string sheet: $\psi_L(\tau), \psi_R(\tau)$ which transform in the fundamental representation of the light flavor group ($SU(2)$ in the present paper). A local hermitean action $S_b = \int d\tau L_f$ is introduced on the boundary $\sigma = 0$ to describe the interaction with background chiral fields $U(x(\tau)) = \exp(i\pi(x)/f_\pi)$ where the normalization scale $f_\pi \simeq 93 \text{ MeV}$, the weak pion decay constant, is introduced to relate the field $\pi(x)$ to a π -meson one.

The boundary Lagrangian is chosen to be reparameterization invariant and in its simplest minimal form reads

$$L_f = \frac{1}{2}i \left(\bar{\psi}_L U(1-z) \dot{\psi}_R - \bar{\psi}_L U(1+z) \dot{\psi}_R + \bar{\psi}_R U^+(1+z^*) \dot{\psi}_L - \bar{\psi}_R U^+(1-z^*) \dot{\psi}_L \right), \quad (5)$$

herein and further on a dot implies a τ derivative: $\dot{\psi} \equiv d\psi/d\tau$. In order to arrive to Eq. (5) a number of field redefinitions have been made. It is impossible to simplify Eq. (5) any further. Details are given in Appendix A.

A further restriction is obtained by requiring CP invariance. There are two CP -like transformations. The first one is

$$U \leftrightarrow U^+, \quad \psi_L \leftrightarrow \psi_R. \quad (6)$$

The above Lagrangian is CP symmetric for $z = -z^* = ia$. The second one is

$$U \leftrightarrow U^+, \quad \psi_L \rightarrow U^+ \psi_L, \quad \psi_R \rightarrow U \psi_R. \quad (7)$$

Under this transformation the Lagrangian becomes invariant only for $z = 0$. We interpret the first CP transformation as the physical one and the one which one should require of a Lagrangian describing strong interactions.

The above coupling may appear surprising at first and somewhat *ad hoc*. To see that this is not so, let us expand the non-linear field $U(x)$, i.e. $U(x) \simeq 1 + i\pi(x)/f_\pi + \dots$ and retain the first two terms. The first term just gives rise to a θ -function propagator which eventually leads to the familiar ordering in the usual string amplitudes $t_1 < t_2 < \dots$. The second term just provides (after integrating the

fermions out) the usual (tachyonic!) vertex. In short, if we ignore the non-linearities in the theory we are back to the usual difficulties.

It is easy to see that the previous action is invariant under general coordinate transformations of the two dimensional world sheet. The fermion action is automatically conformally invariant, because it does not contain the two dimensional world sheet metric tensor since it can be written as a line integral.

In ²³ the cases $z = \pm 1$ were considered. None of them is a valid one on symmetry grounds.

3. Diagrammar

Now we expand $U(x(\tau))$ around a constant background x_0 and look for the potentially divergent one particle irreducible diagrams (1PI). We classify them according to the number of loops. Each additional loop comes with a power of α' .

We expand the function $U(x)$ in powers of the string coordinate field $x_\mu(\tau) = x_{0\mu} + \tilde{x}_\mu(\tau)$ around a constant x_0 which is the translational zero mode of the string

$$\begin{aligned} U(x) &= U(x_0) + \tilde{x}_\mu(\tau) \partial_\mu U(x_0) + \frac{1}{2} \tilde{x}_\mu(\tau) \tilde{x}_\nu(\tau) \partial_\mu \partial_\nu U(x_0) + \dots \\ &\equiv U(x_0) + \mathcal{V}(\tilde{x}). \end{aligned} \quad (8)$$

One can find a resemblance to the familiar derivative expansion of chiral perturbation theory. Indeed perturbation theory in the operators (8) makes sense as a low momentum expansion which is presumably valid up to momenta approaching to the mass of the first massive resonance (ρ meson etc.). In the present case α' is the dimensional parameter normalizing the above expansion.

The free fermion propagator is

$$\langle \psi_R(\tau) \bar{\psi}_L(\tau') \rangle = U^{-1}(x_0) \theta(\tau - \tau'). \quad (9)$$

If we impose CP symmetry then

$$\langle \psi_L(\tau) \bar{\psi}_R(\tau') \rangle = \langle \psi_R(\tau) \bar{\psi}_L(\tau') \rangle^\dagger = U(x_0) \theta(\tau - \tau'), \quad (10)$$

for unitary chiral fields $U(x)$.

The free boson propagator projected on the boundary is

$$\langle x_\mu(\tau) x_\nu(\tau') \rangle = \delta_{\mu\nu} \Delta(\tau - \tau'), \quad \Delta(\tau \rightarrow \tau') = \Delta(0) \sim \frac{\alpha'}{\epsilon}, \quad \partial_\tau \Delta(\tau \rightarrow \tau') = 0, \quad (11)$$

the latter results hold in dimensional regularization (see below).

In order to make contact between dimensional regularization, a short-distance cut-off (which we shall later use) and Regge phenomenology we need to unambiguously fix the normalization of the string propagator. This can be inferred from the definition of the kernel of the N-point tachyon amplitude for the open string²⁸. The Veneziano amplitude corresponds to the insertion of vertex operators

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: $\exp(ik_\mu^{(j)}x^\mu(\tau_j))$: on the boundary of the string. After resolution of the Gaussian integral one obtains for the kernel of the generalized beta-function

$$\begin{aligned} \langle \prod_j : \exp(ik_\mu^{(j)}x_\mu(\tau)) : \rangle &= \exp \left(-\frac{1}{2} \sum_{j \neq l} k_\mu^{(j)} k_\mu^{(l)} \Delta(\tau_j - \tau_l) \right) \\ &\equiv \prod_{j>l} |\tau_j - \tau_l|^{2\alpha' k^{(j)} k^{(l)}}, \end{aligned} \quad (12)$$

which unambiguously prescribes

$$\Delta(\tau_j - \tau_l) = -2\alpha' \ln(|\tau_j - \tau_l| \mu). \quad (13)$$

The μ dependence does not show up in (12) due to energy-momentum conservation.

Keeping in mind this definition let us determine the string propagator in dimensional regularization, restricted on the boundary. First we calculate the momentum integral in $2 + \epsilon$ dimensions

$$\begin{aligned} \Delta_\epsilon(\tau) &= 4\pi\alpha' \left(\frac{\varphi}{\mu} \right)^\epsilon \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{\exp(ik_0\tau)}{k^2} \\ &= \alpha' \Gamma \left(\frac{\epsilon}{2} \right) \left| \frac{\tau\mu\sqrt{\pi}}{\varphi} \right|^{-\epsilon} \\ &\stackrel{\epsilon \rightarrow 0}{\equiv} 2\alpha' \left[\frac{1}{\epsilon} + C - \ln \left(\frac{\tau\mu}{\varphi} \right) \right] + \mathcal{O}(\epsilon). \end{aligned} \quad (14)$$

A dimensionally regularized propagator properly normalized to reproduce (13) can be constructed by subtracting from (14) its value at $\tau\mu = 1$ where (13) should vanish

$$\begin{aligned} \Delta_\epsilon(\tau)|_{reg} &= \alpha' \Gamma \left(\frac{\epsilon}{2} \right) \left\{ \left| \frac{\tau\mu\sqrt{\pi}}{\varphi} \right|^{-\epsilon} - \left| \frac{\sqrt{\pi}}{\varphi} \right|^{-\epsilon} \right\} \\ &\stackrel{\epsilon \rightarrow 0}{\equiv} -2\alpha' \ln |\tau\mu| + \mathcal{O}(\epsilon). \end{aligned} \quad (15)$$

Therefrom one finds unambiguously the relation

$$\Delta(0) = -\alpha' \Gamma \left(\frac{\epsilon}{2} \right) \left| \frac{\sqrt{\pi}}{\varphi} \right|^{-\epsilon} \stackrel{\epsilon \rightarrow 0}{\equiv} -2\alpha' \left[\frac{1}{\epsilon} + C + \ln \varphi \right] + \mathcal{O}(\epsilon) \equiv \Delta_\epsilon - 2\alpha' \ln \varphi, \quad (16)$$

where in the spirit of dimensional regularization we have assumed that $\epsilon < 0$ and hence the first term in (15) vanishes at $\tau = 0$.

The two-fermion, N -boson vertex operators are generated by the expansion (8) and they appear with an extra sign $(-)^i = i^2$ following the definition for the generating functional $Z_b = \langle \exp(iS_b) \rangle$ and Eq. (5). In particular, for the $L \rightarrow R$ transition one has

$$V = -\frac{1}{2} ((1-z)\mathcal{V}(\tilde{x})\partial_\tau + (1+z)\partial_\tau [\mathcal{V}(\tilde{x}) \dots]), \quad (17)$$

and for the $R \rightarrow L$ transition the hermitean conjugated vertex V^+ appears. The corresponding Feynman rule for the 2-fermion, N-boson vertex of the $L \rightarrow R$ transition is

$$\mathcal{V}_N = -\frac{1}{2n!} \partial_A \delta(A - B) \partial_{\mu_1} \dots \partial_{\mu_N} U(x_0) [(1 - z) \delta(A - \tau_1) \dots \delta(A - \tau_N) + (1 + z) \delta(\tau_1 - B) \dots \delta(\tau_N - B)] \quad (18)$$

where A, B are proper-time values for the left- and right-handed fermions and τ_1, \dots, τ_N are proper-time values for the boson field-string variables.

From this point on it is quite straightforward to proceed with the renormalization process. We shall determine the counterterms required to make the beta functional for the coupling $U(x)$ vanish up to the two loop level. In spite of the relative complexity of the Feynman rules, the fact that we are working with a boundary field theory is crucial in making the calculation manageable. In fact most diagrams can be determined by simply playing with integration by parts and using basic properties of the Dirac delta function. Yet the renormalization is quite non-trivial and the ultraviolet structure of the counterterms is surprisingly quite complex. It is thanks to this complexity that non-zero values for the $O(p^4)$ coefficients can be obtained. In fact we believe that some of the results presented in this work can have some bearing on more general discussions involving fundamental strings too.

In what follows we retain not only the singular parts of one-loop diagrams but also the finite ones as they will be necessary to construct two-loop diagrams.

4. Renormalization of the fermion propagator at one loop

To avoid clutter the main body of the paper we have relegated the detailed derivation of the different Feynman diagrams to the appendixes. Since the present calculation is somewhat non-standard, we provide the technical details there.

Using the set of Feynman rules described in the previous section one arrives to the following result for the divergent part of the propagator (Appendix B),

$$\theta(A - B) \frac{1}{2} \Delta(0) U^{-1} \left\{ -\partial_\mu^2 U + \frac{3 + z^2}{2} \partial_\mu U U^{-1} \partial_\mu U \right\} U^{-1}. \quad (19)$$

This divergence is eliminated by introducing an appropriate counterterm $U \rightarrow U + \delta U$

$$\delta U = \Delta(0) \left[\frac{1}{2} \partial_\mu^2 U - \frac{3 + z^2}{4} \partial_\mu U U^{-1} \partial_\mu U \right] = 0. \quad (20)$$

Conformal symmetry is restored (the beta-function is zero) if the above contribution vanishes.

Let us find out for which value of z this variation of U is compatible with its unitarity.

$$\delta(UU^+) = U \cdot \delta U^+ + \delta U \cdot U^+ = 0. \quad (21)$$

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A simple calculation shows that this takes place for $z = \pm i$. For other values of z eq.(20) entails $\partial_\mu U U^{-1} \partial_\mu U = 0$ which has only a trivial constant solution in Euclidean space-time. In ²³ this value of z was not considered and thus unitarity of U was not properly taken into account.

When $z = \pm i$ and before the unitarity constraints are imposed the local classical action which has (20) as equation of motion is

$$W^{(2)} = \frac{f_\pi^2}{8} \int d^4x \text{tr} [\partial_\mu U \partial_\mu U^{-1} + \partial_\mu U^+ \partial_\mu (U^+)^{-1}]. \quad (22)$$

For other values of $z \neq \pm 1; \pm i$ the related local action is unknown. The above Lagrangian is of course the well known non-linear sigma model which is commonly employed to describe pion interactions.

We have thus succeeded in finding the action induced by the QCD string. It has all the required properties of locality, chiral symmetry and proper low momentum behavior (Adler zero). Furthermore, it describes massless pions. f_π , the overall normalization scale, cannot be predicted from these arguments.

To this point we have been quite successful in our program, but of course no real predictability has been achieved yet. Indeed we knew the form of this action from general principles, even though it is nice to see that things work out consistently. We have to turn to the $O(p^4)$ effective Lagrangian to get non-universal results.

5. Renormalization of the vertices at one loop

In order to proceed to a two loop calculation we shall need in addition to the counterterms for the one-loop propagator (which we just got) the counterterms for the vertex with two fermion lines and one and two boson lines, x^μ , respectively. We also have to check whether the minimal Lagrangian (5) is sufficient to renormalize also the vertices. It turns out that, in fact, it is not.

Let us obtain the divergences for vertices with external boson lines. We introduce an external background boson field \bar{x}_μ to describe vertices with several boson legs and split $x_\mu = \bar{x}_\mu + \eta_\mu$. The free propagator for the fluctuation field η_μ coincides with the one for x_μ .

Then the total divergence in the vertex with two fermions and one boson line is (see Appendix C)

$$\begin{aligned} & \theta(A - B) \frac{1}{4} \Delta(0) U^{-1} \left\{ \bar{x}_\mu(A) (1 + z) \left[-\partial_\mu (\partial^2 U) + 2 \partial_\nu U U^{-1} \partial_\mu \partial_\nu U \right. \right. \\ & \quad \left. \left. + (1 + z) \partial_\mu \partial_\nu U U^{-1} \partial_\nu U - \frac{1}{2} (1 + z) (3 - z) \partial_\nu U U^{-1} \partial_\mu U U^{-1} \partial_\nu U \right] \right. \\ & \quad \left. + \bar{x}_\mu(B) (1 - z) \left[-\partial_\mu (\partial^2 U) + (1 - z) \partial_\nu U U^{-1} \partial_\mu \partial_\nu U + 2 \partial_\mu \partial_\nu U U^{-1} \partial_\nu U \right. \right. \\ & \quad \left. \left. - \frac{1}{2} (1 - z) (3 + z) \partial_\nu U U^{-1} \partial_\mu U U^{-1} \partial_\nu U \right] \right\} U^{-1} \\ & \equiv -\frac{1}{2} \theta(A - B) U^{-1} \left[\bar{x}_\mu(B) \Phi_\mu^{(1)} + \bar{x}_\mu(A) \Phi_\mu^{(2)} \right] U^{-1}. \end{aligned} \quad (23)$$

Let us now amputate the fermion legs and evaluate the pure vertex divergences. The amputation rules are

$$\begin{aligned}\bar{x}_\mu(A)\theta(A-B) &= -\int d\tau\partial_\tau\theta(A-\tau)\bar{x}_\mu(\tau)\theta(\tau-B), \\ \bar{x}_\mu(B)\theta(A-B) &= \int d\tau\theta(A-\tau)\bar{x}_\mu(\tau)\partial_\tau\theta(\tau-B).\end{aligned}\quad (24)$$

Then the divergent part (23) can be reproduced by the following operator in the Lagrangian

$$\frac{i}{2}\left(\bar{\psi}_L\Phi^{(1)}\dot{\psi}_R - \bar{\psi}_L\Phi^{(2)}\dot{\psi}_R\right) + \text{h.c.}, \quad \Phi^{(1,2)} \equiv \bar{x}_\mu(\tau)\Phi_\mu^{(1,2)}, \quad (25)$$

where the vertex matrices $\Phi_\mu^{(1,2)}$ can be rearranged as follows

$$\begin{aligned}\Phi_\mu^{(1)} &= \Delta(0)\left\{(1-z)\partial_\mu\left(\frac{1}{2}\partial^2U - \frac{3+z^2}{4}\partial_\nu UU^{-1}\partial_\nu U\right) \right. \\ &\quad \left. - \frac{1-z^2}{2}\left(\frac{1-z}{2}\partial_\mu\partial_\nu UU^{-1}\partial_\nu U - \frac{1+z}{2}\partial_\nu UU^{-1}\partial_\mu\partial_\nu U\right) \right. \\ &\quad \left. + z\partial_\nu UU^{-1}\partial_\mu UU^{-1}\partial_\nu U\right\} \\ &\equiv (1-z)\partial_\mu(\delta U) - \phi_\mu \\ \Phi_\mu^{(2)} &= (1+z)\partial_\mu(\delta U) + \phi_\mu.\end{aligned}\quad (26)$$

The terms proportional to derivatives of δU are automatically eliminated by the redefinition of U that one performs to renormalize the one-loop propagator (and it of course vanishes if the equations of motion are imposed). But the part proportional to ϕ_μ remains and to absorb these divergences new counterterms are required. Evidently, these terms come out of the following terms in the Lagrangian:

$$\begin{aligned}\Delta L_{div.} &= \frac{i}{4}\Delta(0)(1-z^2)\bar{\psi}_L\left(\frac{1-z}{2}\partial_\nu\dot{U}U^{-1}\partial_\nu U - \frac{1+z}{2}\partial_\nu UU^{-1}\partial_\nu\dot{U}\right. \\ &\quad \left.+ z\partial_\nu UU^{-1}\dot{U}U^{-1}\partial_\nu U\right)\psi_R + \text{h.c.}\end{aligned}\quad (27)$$

Therefore the counterterms required to eliminate the additional divergences for the vertex with one boson and two fermion lines can be parameterized with three bare constants g_1 , g_2 and g_3 , which are real if the CP symmetry for $z = -z^*$ holds

$$\begin{aligned}\Delta L_{bare} &= \frac{i}{8}(1-z^2)\bar{\psi}_L\left((g_1 - zg_2)\partial_\nu\dot{U}U^{-1}\partial_\nu U - (g_1 + zg_2)\partial_\nu UU^{-1}\partial_\nu\dot{U}\right. \\ &\quad \left.+ 2zg_3\partial_\nu UU^{-1}\dot{U}U^{-1}\partial_\nu U\right)\psi_R + \text{h.c.}\end{aligned}\quad (28)$$

Renormalization is accomplished by redefining the couplings g_i in the following way

$$g_i = g_{i,r} - \Delta(0). \quad (29)$$

The constants $g_{i,r}$ are finite, but in principle scheme dependent, and from (16) it follows that a logarithmic dependence of the bare couplings on the conformal factor φ is introduced along the renormalization process. The counterterms are of higher

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dimensionality than the original Lagrangian (5) and therefore the couplings g_i are of dimension M^{-2} . Since (5) was actually the most general coupling permitted by the symmetries of the model, in particular conformal invariance, one is lead to the conclusion that conformal symmetry is broken, already at tree level, by these couplings, unless they happen to vanish. Since they are dimensional, it is natural to normalize them by the only dimensional parameter, namely α' .

Even if the new couplings are dimensional, it turns out that at the order we are computing, the trace of the energy-momentum tensor is still vanishing once the requirements of unitarity of U are taken into account (see Appendix D) and therefore conformal invariance is not broken at the order we are working. At higher orders in the α' expansion further counterterms may be however required in order to ensure conformal invariance perturbatively. We postpone a more detailed discussion to the final sections.

One can introduce the running “effective” couplings

$$g_i + \Delta_\epsilon = g_{i,r} + 2\alpha' \ln \varphi \equiv g_i^\varphi, \quad (30)$$

One-loop conformal invariants are $g_1^\varphi - g_2^\varphi$ and $g_1^\varphi - g_3^\varphi$. At any rate, the dependence of the new couplings on the Liouville mode is determined.

In any case, the appearance of new vertices changes the fermion propagator due to the diagrams discussed in Appendix E. One obtains from such terms (which are of higher order in derivatives) the following contribution to the propagator

$$\begin{aligned} & \theta(A - B) \frac{1}{16} \Delta(0) (1 - z^2) U^{-1} \{ 2(g_{1,r} - z^2 g_{2,r}) \partial_\rho U U^{-1} \partial_\mu \partial_\rho U U^{-1} \partial_\mu U \\ & - (1 + z)(g_{1,r} + z g_{2,r}) \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho \partial_\mu U \\ & - (1 - z)(g_{1,r} - z g_{2,r}) \partial_\rho \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu U \\ & + 4z^2 g_{3,r} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu U \} U^{-1} \\ & \equiv -\theta(A - B) \Delta(0) U^{-1} \delta^{(4)} U U^{-1} \end{aligned} \quad (31)$$

we shall denote this contribution by $\theta(A - B)d_g$. One should add the divergence contained in d_g to the one-loop result, thereby modifying the U field renormalization and equations of motion

$$\bar{\delta}U = \Delta(0) \left[\frac{1}{2} \partial_\mu^2 U - \frac{3 + z^2}{4} \partial_\mu U U^{-1} \partial_\mu U + \delta^{(4)} U \right] = 0. \quad (32)$$

This is, in fact, the source of the much sought after $O(p^4)$ terms.

We note that this new contribution (31) is proportional to $1 - z^2$ and it was thus absent in ²³. At this point we have to ask whether such equation of motion may be derived from a local effective Lagrangian containing both dimension 2 and dimension 4 operators. This would then constitute the effective Lagrangian derived from the string model. However at this point this question is too premature to formulate. Two-loop diagrams generated from (5) could certainly produce similar contributions and, as a matter of fact, so could new counterterms from diagrams

with two fermions and two boson lines, should they require an additional counterterm. In fact it can be seen that the above equation of motion cannot be derived from a local Lagrangian involving the unitary matrix U . The requirements of locality and unitarity would force $g_{j,r} = 0$, so this should not be the full answer.

Before concluding this section and moving to the other contributions we have just mentioned, we calculate the renormalization of the vertex with two fermion and two boson lines, as this is also required as a counterterm. Let us summarize the divergent structure for these diagrams (see Appendix F)

$$\begin{aligned}
& \theta(A - B) \frac{1}{4} U^{-1} \{ \bar{x}_\mu(A) \bar{x}_\nu(A) [\partial_\mu \partial_\nu (-\delta U)(1+z) - \phi_{\mu\nu}] \\
& + \bar{x}_\mu(B) \bar{x}_\nu(B) [\partial_\mu \partial_\nu (-\delta U)(1-z) + \phi_{\mu\nu}] \\
& + \Delta(0) \text{frac} 1 - z^2 2 \int_B^A d\tau [-\bar{x}_\mu(\tau) \dot{\bar{x}}_\nu(\tau) \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\nu \partial_\rho U (1-z) \\
& + \dot{\bar{x}}_\mu(\tau) \bar{x}_\nu(\tau) \partial_\rho \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\rho U (1+z) \\
& + (\bar{x}_\mu(\tau) \dot{\bar{x}}_\nu(\tau) - \dot{\bar{x}}_\mu(\tau) \bar{x}_\nu(\tau)) \partial_\rho \partial_\mu U U^{-1} \partial_\nu \partial_\rho U \} U^{-1}, \tag{33}
\end{aligned}$$

where

$$\begin{aligned}
\phi_{\mu\nu} = \Delta(0) \frac{1-z^2}{2} & \left[\frac{1-z}{2} \partial_\rho \partial_\mu \partial_\nu U U^{-1} \partial_\rho U - \frac{1+z}{2} \partial_\rho U U^{-1} \partial_\mu \partial_\nu \partial_\rho U \right. \\
& - z \partial_\rho \partial_\mu U U^{-1} \partial_\nu \partial_\rho U + z \partial_\rho U U^{-1} \partial_\mu \partial_\nu U U^{-1} \partial_\rho U \\
& + 2z (\partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\nu \partial_\rho U + \partial_\rho \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\rho U) \\
& \left. - 2z \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\rho U \right] \tag{34}
\end{aligned}$$

One can check that the terms encoded in $\phi_{\mu\nu}$, together with the last contribution in Eq. (33) combine precisely as a second variation of the additional interaction vertices (27) in coordinate fields. Therefore their renormalization is completely performed with the help of counterterms (28) and no additional counterterms or operators appear.

It can also be seen that, in fact, any diagram with an arbitrary number of external boson lines and two fermion lines, i.e. any vertex of those generated by the perturbative expansion of (5) is rendered finite by the previous counterterms. This completes the renormalization program at one loop.

6. The fermion propagator at two loops

The two-loop contributions to the fermion propagator can be obtained from one-loop diagrams with two external boson legs by joining the latter with a boson propagator. A factor $1/2$ has to be added. One must include not only one-particle irreducible diagrams but also one-particle reducible ones and consider both divergent and finite parts.

There are 10 two-loop diagrams which are listed in Appendix G. The divergences in the propagator at two-loops can be presented separated into five pieces

$$\theta(A - B) [d_I + d_{II} + d_{III} + d_{IV} + d_V]. \tag{35}$$

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The first and second piece contain the double divergence $\Delta^2(0)$, the third, fourth and fifth pieces reveal only a single divergence $\Delta(0)$. Finite parts are irrelevant for the present discussion.

Table 1. Chiral field structures proportional to the double divergence $\Delta^2(0)$ appearing in the two loop contribution to the fermion propagator.

| CF structure | d_I | d_{II} | Total |
|---------------------------|---|------------------------------|-----------------------------|
| $\mu^2 \rho^2$ | $-\frac{1}{8z^2}$ | 0 | $-\frac{1}{8z^2}$ |
| $\mu^2 \rho - \rho$ | $\frac{3+z^2}{8}$ | 0 | $\frac{3+z^2}{8}$ |
| $\rho - \mu^2 \rho$ | $\frac{3+z^2}{8}$ | 0 | $\frac{3+z^2}{8}$ |
| $\rho - \mu^2 - \rho$ | $-\frac{3+z^2}{8}$ | 0 | $-\frac{3+z^2}{8}$ |
| $\rho - \mu - \mu \rho$ | $-\frac{3+z^2}{8} - \frac{(3+z^2)^2}{32}$ | $-\frac{(1-z^2)(1+z)^2}{32}$ | $-\frac{11+z+5z^2-z^3}{16}$ |
| $\mu \rho - \mu - \rho$ | $-\frac{3+z^2}{8} - \frac{(3+z^2)^2}{32}$ | $-\frac{(1-z^2)(1-z)^2}{32}$ | $-\frac{11-z+5z^2+z^3}{16}$ |
| $\mu \rho - \mu \rho$ | $\frac{3+z^2}{8}$ | 0 | $\frac{3+z^2}{8}$ |
| $\rho - \mu - \mu - \rho$ | $\frac{3+z^2}{8} + \frac{(3+z^2)^2}{32}$ | 0 | $\frac{(3+z^2)(7+z^2)}{32}$ |
| $\mu - \mu \rho - \rho$ | $-\frac{(3+z^2)^2}{16}$ | $\frac{(1-z^2)^2}{8}$ | $-\frac{1+z^2}{8}$ |
| $\rho - \mu - \rho - \mu$ | $\frac{(3+z^2)^2}{16}$ | $\frac{(1-z^2)^2 z^2}{8}$ | $\frac{(9-z^2)(1+z^2)}{16}$ |

The component d_I represents “the second variation”, or one-loop divergence in the one-loop divergence

$$\begin{aligned}
 d_I &= -\frac{1}{2}U^{-1}\delta(\delta U)U^{-1} = -\frac{1}{2}\Delta(0)U^{-1} \left\{ \frac{1}{2}\partial_\mu^2(\delta U) - \right. \\
 &\quad \left. -\frac{3+z^2}{4} [\partial_\mu(\delta U)U^{-1}\partial_\mu U - \partial_\mu U U^{-1}\delta U U^{-1}\partial_\mu U + \partial_\mu U U^{-1}\partial_\mu(\delta U)] \right\} U^{-1}, \\
 \delta U &= \Delta(0) \left[\frac{1}{2}\partial_\mu^2 U - \frac{3+z^2}{4}\partial_\mu U U^{-1}\partial_\mu U \right]. \tag{36}
 \end{aligned}$$

Therefore it is renormalized away by the redefinition of the U field and vanishes when the equations of motion are imposed. The counterterms renormalizing U field yield the same expression but twice more and of the opposite sign. Thus the result is $-d_I$ in correspondence with the results of ²³ (for $z = 1$ they coincide). This is all that was obtained in that work. In particular no single-pole, $\Delta(0)$ appeared and therefore no new equations were obtained at the two loop level. Accordingly, the coefficients of the $O(p^4)$ coefficients were deemed to vanish. This will not be the case here.

The second part represents the remaining terms of order $\Delta^2(0)$ in two loop diagrams after subtraction of d_I and it reads

$$\begin{aligned}
 d_{II} &= \frac{1}{32}\Delta^2(0)(1-z^2)U^{-1} \left\{ 2(1-z^2)\partial_\rho U U^{-1}\partial_\mu \partial_\rho U U^{-1}\partial_\mu U \right. \\
 &\quad \left. - (1+z)^2\partial_\rho U U^{-1}\partial_\mu U U^{-1}\partial_\rho \partial_\mu U - (1-z)^2\partial_\rho \partial_\mu U U^{-1}\partial_\rho U U^{-1}\partial_\mu U \right. \\
 &\quad \left. + 4z^2\partial_\rho U U^{-1}\partial_\mu U U^{-1}\partial_\rho U U^{-1}\partial_\mu U \right\} U^{-1}. \tag{37}
 \end{aligned}$$

This term is identical, but of opposite sign, to the contributions generated by the one-loop counterterm in the vertex with two fermions and one boson line, after its insertion in a one-loop diagram (see Appendix E).

To summarize, we show in Table 1 the distribution of $\Delta^2(0)$ divergences between d_I and d_{II} . Short-hand notations are used for the corresponding chiral field (CF) operators, for instance, $\mu - \mu\rho - \rho$ corresponds to $U^{-1}\partial_\mu U U^{-1}\partial_\mu\partial_\rho U U^{-1}\partial_\rho U U^{-1}$.

In d_{III} we include those single-pole divergences, proportional to $\Delta(0)$, which are removed once the one-loop renormalization of U in the finite nonlocal part of fermion propagator at one loop (see, Eq. (B.2)) is taken into account, that is when we replace U by $U + \delta^{(2)}U$ in the one-loop propagator

$$\begin{aligned} & \frac{1}{4}(1-z^2)\Delta(0)\Delta(A, B)U^{-1} [\partial_\mu(\delta U)U^{-1}\partial_\mu U \\ & - \partial_\mu U U^{-1}\delta U U^{-1}\partial_\mu U + \partial_\mu U U^{-1}\partial_\mu(\delta U)] U^{-1}. \end{aligned} \quad (38)$$

Likewise in d_{IV} we include the divergences that are eliminated when the additional counterterms in the one-boson vertices (those proportional to g_i) are included in the finite part of the one-loop fermion propagator (the terms proportional to $\Delta(A, B)$ in Eq. (E.1)).

One can check that all terms in the two loop fermion propagator linear in $\Delta(0)$ and in $\Delta(A, B)$ belong either to d_{III} or to d_{IV} . We present them in Table 2. Thus one-loop renormalization removes d_{III} and d_{IV} completely.

Table 2. Summary of single divergences $\Delta(0)$ which are eliminated after the introduction of the one-loop counterterms.

| CF structure | d_{III} | $d_{IV} \leftrightarrow -2d_{II}$ | Total |
|---------------------------|------------------------------|-----------------------------------|-----------------------------|
| $\mu^2\rho - \rho$ | $\frac{1-z^2}{8}$ | 0 | $\frac{1-z^2}{8}$ |
| $\rho - \mu^2\rho$ | $\frac{1-z^2}{8}$ | 0 | $\frac{1-z^2}{8}$ |
| $\rho - \mu^2 - \rho$ | $-\frac{1-z^2}{8}$ | 0 | $-\frac{1-z^2}{8}$ |
| $\rho - \mu - \mu\rho$ | $-\frac{(1-z^2)(3+z^2)}{16}$ | $\frac{(1-z^2)(1+z)^2}{16}$ | $-\frac{(1-z^2)(1-z)}{8}$ |
| $\mu\rho - \mu - \rho$ | $-\frac{(1-z^2)(3+z^2)}{16}$ | $\frac{(1-z^2)(1-z)^2}{16}$ | $-\frac{(1-z^2)(1+z)}{8}$ |
| $\mu\rho - \mu\rho$ | 0 | 0 | 0 |
| $\rho - \mu - \mu - \rho$ | $\frac{(1-z^2)(3+z^2)}{16}$ | 0 | $\frac{(1-z^2)(3+z^2)}{16}$ |
| $\mu - \mu\rho - \rho$ | $-\frac{(1-z^2)(3+z^2)}{16}$ | $-\frac{(1-z^2)^2}{8}$ | $-\frac{1-z^2}{8}$ |
| $\rho - \mu - \rho - \mu$ | $\frac{(1-z^2)(3+z^2)}{8}$ | $-\frac{(1-z^2)^2 z^2}{4}$ | $\frac{(1-z^2)(3-z^2)}{8}$ |

Some single-pole divergences remain however. Indeed, there are some divergences linear in $\Delta(0)$ which come from the double integral in the diagrams of Appendix G, (G.8) and (G.10),

$$J(A, B) = \int_B^A d\tau_1 \int_B^{\tau_1} d\tau_2 \partial_{\tau_1} \Delta(\tau_1 - \tau_2) \partial_{\tau_2} \Delta(\tau_1 - \tau_2)$$

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$$= - \int_0^{A-B} d\tau (A - B - \tau) [\dot{\Delta}(\tau)]^2. \quad (39)$$

In Appendix H this integral is calculated using two different regularizations. The divergence is found to be

$$\begin{aligned} d_V &= c_V \Delta(0) [U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\mu U U^{-1} \partial_\rho U U^{-1} \\ &\quad - U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1}] \\ &\equiv -\Delta(0) U^{-1} \bar{d}_V U^{-1}. \end{aligned} \quad (40)$$

with $c_V = \alpha'(1 - z^2)^2/8 = \alpha'/2$ for $z = \pm i$. This term survives after adding all the counterterms and together with (31) are the only new genuine divergences that can contribute to the beta function (single poles). It must therefore be added to the equation of motion at the next order in the α' expansion and modifies the term $\delta^{(4)}U$, $\delta^{(4)}U \rightarrow \delta^{(4)}U + \bar{d}_V$ in a crucial manner; namely it opens the way to non zero solutions for the coupling constants g_i and therefore for nonzero values for the Gasser-Leutwyler $O(p^4)$ coefficients.

7. Local integrability and unitarity

The equation of motion of $O(p^2)$, Eq. (20), can be obtained from a local action of the Weinberg type (22), involving a unitary matrix $U(x)$, only for $z = \pm i$. If the corresponding terms with four derivatives that we have just found are to be derived from dimension-four operators in a local effective Lagrangian then certain relations are for sure to be required from the so far arbitrary constants $g_{i,r}$.

Such a Lagrangian has only two terms compatible with the chiral symmetry if we use the dimension-two equations of motion (20),

$$\mathcal{L}^{(4)} = \frac{1}{2} f_\pi^2 \text{tr} (K_1 \partial_\mu U \partial_\rho U^{-1} \partial_\mu U \partial_\rho U^{-1} + K_2 \partial_\mu U \partial_\mu U^{-1} \partial_\rho U \partial_\rho U^{-1} + h.c.). \quad (41)$$

The terms

$$\partial_\mu^2 U \partial_\rho U^{-1} \partial_\rho U U^{-1}, \quad \partial_\mu^2 U \partial_\rho^2 U^{-1}, \quad (\partial_\mu^2)^2 U U^{-1}, \quad \partial_\mu \partial_\rho U \partial_\mu \partial_\rho U^{-1}$$

which are in principle possible are reduced to the set (41) with the help of integration by parts in the action and of the dimension-two equations of motion (20).

Variation of the previous Lagrangian gives the following addition to the equations of motion

$$\begin{aligned} \frac{\delta S}{\delta U} &= -f_\pi^2 U^{-1} \{ 2K_1 [\partial_\mu \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho U + \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu \partial_\rho U \\ &\quad - \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho U - 2\partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\rho U U^{-1} \partial_\mu U \\ &\quad + \partial_\mu U U^{-1} \partial_\rho^2 U U^{-1} \partial_\mu U] \\ &\quad + K_2 [\partial_\mu \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho U + \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu \partial_\rho U + \\ &\quad 2\partial_\mu U U^{-1} \partial_\mu \partial_\rho U U^{-1} \partial_\rho U \\ &\quad + \partial_\mu^2 U U^{-1} \partial_\rho U U^{-1} \partial_\rho U + \partial_\mu U U^{-1} \partial_\mu U^{-1} \partial_\rho^2 U \end{aligned}$$

$$\begin{aligned}
& -\partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\rho U U^{-1} \partial_\mu U - 2\partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho U \\
& - 3\partial_\mu U U^{-1} \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\rho U \} U^{-1}.
\end{aligned} \tag{42}$$

Table 3. Comparison between the coefficients of the different chiral field structures in the equations of motion derived from a local Lagrangian and from the condition of vanishing beta function, namely vanishing of the single pole divergences at the two loop level

| CF structure | χ -lagr. | d_g | d_V |
|------------------------------|-----------------------|---|--------|
| $\mu - -\rho - -\mu\rho$ | $-2(2K_1 + K_2)$ | $\frac{1}{16}(1-z^2)(1+z)(g_{1,r} + z g_{2,r})$ | 0 |
| $\mu\rho - -\mu - -\rho$ | $-2(2K_1 + K_2)$ | $\frac{1}{16}(1-z^2)(1-z)(g_{1,r} - z g_{2,r})$ | 0 |
| $\mu - -\mu\rho - -\rho$ | $-4K_2$ | $\frac{1}{8}(1-z^2)(-g_{1,r} + z^2 g_{2,r})$ | 0 |
| $\mu - -\rho - -\rho - -\mu$ | $2[(1-z^2)K_1 + K_2]$ | 0 | $-c_V$ |
| $\mu - -\mu - -\rho - -\rho$ | $-2z^2 K_2$ | 0 | 0 |
| $\mu - -\rho - -\mu - -\rho$ | $4[K_1 + K_2]$ | $-\frac{1}{4}(1-z^2)z^2 g_{3,r}$ | c_V |

Let us now apply the $O(p^2)$ equations of motion to remove the Laplacian on chiral fields $\partial_\mu^2 U$. Then one obtains the set of coefficients for the various chiral field structures given in Table 3. These coefficients to be determined are then compared with the results obtained from the coefficients of the one-loop and two-loop single-pole divergences (see (31) and (40)). For $z^2 = -1$ only one solution is possible, implying

$$K_2 = 0, \quad K_1 = -\frac{1}{4}c_V = -\frac{\alpha'}{8}; \quad g_{1,r} = -g_{2,r} = -g_{3,r} = 4c_V. \tag{43}$$

Thus, comparing Eq. (41) with the usual parameterization of the Gasser and Leutwyler Lagrangian³⁰,

$$L_1 = \frac{1}{2}L_2 = -\frac{1}{4}L_3 = -\frac{1}{2}K_1 f_\pi^2 = \frac{f_\pi^2 \alpha'}{16}. \tag{44}$$

For $\alpha' = 0.9 \text{ GeV}^{-2}$ and $f_\pi \simeq 93 \text{ MeV}$ it yields $L_2 \simeq 0.9 \cdot 10^{-3}$ which is quite a satisfactory result³¹.

So far we have paid no attention to the unitarity of U at the two-loop level. Does the variation implied by d_g and d_V respect the perturbative unitarity of U ? Or does this lead to a new constraint eventually incompatible with the numerical previous numerical values? It turns out (see Appendix I) that if one accepts arbitrary real coefficients in the set of dimension-four vertices included in (42) then the only solution compatible with the unitarity is given by the parameterization with constants K_1 and K_2 . Thus the requirement to preserve unitarity under field renormalization is entirely equivalent to the local integrability condition, similarly to the case of dimension-two operators. This is a remarkable result that hints to the consistency of the whole procedure.

8. Conclusions: conformal invariance and all that

Conformal invariance is a subtle issue. As we have discussed in detail in the introduction this is a necessary ingredient for the consistency of string theory and it is, according to general principles of field theory, equivalent to requiring the vanishing of the beta functions.

There are several sources of breaking of conformal invariance in the present model. We have been concerned with the equations of motion for the $U(x)$ string field functional. The beta functional has been computed up to two loops, properly including the unitarity constraints. The vanishing of the single pole divergences at one loop leads to the familiar non-linear sigma model. At two loops the $O(p^4)$ terms of Gasser and Leutwyler appear. The numerical values turns out to be identically determined by the arguments of locality and unitarity and they therefore determine the ‘renormalized’ value of $g_{i,r}$ in terms of the constant c_V . The latter being the coefficient of a single loop divergence in an integral appears to be manifestly universal. The former could be in principle ambiguous since a change in the renormalization scheme (for instance by choosing the subtraction to a value different from $\mu\tau = 1$ in the x -propagator) may shift a finite piece between $\Delta(0)$ and the renormalized value of the coupling.

However, this ambiguity is of no relevance. The requirements of locality and unitarity seem to restrict the ‘measurable’ value of $g_{i,r}$ to specific, well determined values. If one changes the renormalization scheme by a finite amount δ , the ‘measurable’ couplings will all change by the same amount to $\bar{g}_{i,r} = g_{i,r} + \delta$, but it is this last quantity the one that will then appear in the equations of motion and will be related to c_V , which is universal, so this change of scheme is immaterial.

Furthermore, the renormalized couplings $g_{i,r}$ are conformal factor independent, so the relations determined in the previous section hold in any conformal frame and so are the values for the $O(p^4)$ coefficients of the chiral Lagrangian. This is of course in full agreement with general considerations based in QCD and chiral Lagrangians (in the large N limit, that is, where the string picture is supposed to hold).

In ²³ a general, but tentative, argument (based in conformal invariance) was given that, if valid, would imply the vanishing of the $O(p^4)$ coefficients to all orders in the inverse string tension. Here the argument simply fails because the additional couplings which are required to renormalize the model are dimensional. These couplings were not considered in ²³ because the need for additional counterterms was not manifest along the renormalization process due to the (incorrect) choice of the value of z .

Of course this immediately raises a new issue. The new counterterms seem to entail a breakdown of ‘classical’ conformal invariance. Indeed, introducing the coupling g_i , which are of dimension M^{-2} makes the action non-conformally invariant already at the classical level, in a way that is similar to the introduction of a coupling to the tachyon in the familiar bosonic string theory. Although the issue is

somewhat independent of the equations of motion for the matrix field $U(x)$ we have to give an answer to this difficulty if conformal invariance is to hold.

To demand conformal invariance in the string-mode sector, we have to guarantee the vanishing of the trace of the energy momentum tensor. This issue can be discussed in several ways, but let us proceed to determine the trace of the energy momentum tensor. On general grounds we can write

$$\Theta = \sum_i \beta_i O_i, \quad (45)$$

where β_i is the beta function for the i -th coupling and O_i the accompanying operator. Then, when computing the trace of the energy momentum tensor, the part of the tensor bilinear in fermion fields leads to a contribution that contains operators of the form (28) with their couplings g_i replaced by the appropriate beta functions. These beta functions contain a classical, purely engineering, part as well as a quantum one, which has been computed in the text, and which is the same for all couplings g_i . In the string-mode sector their contribution is given by the averaging over the fermion vacuum and it vanishes according to the arguments presented in the Appendix D. There is no a fermion induced string action when the CP symmetry (6) holds. Thus the presence of conformally non-invariant interaction on the boundary does not affect conformal symmetry in the bulk.

Throughout this paper we have systematically used conventional perturbation theory and the fermion loop expansion which however may be more cumbersome when one proceeds to the vertices with larger number of emitted bosons and to higher orders in the loop expansion. As an alternative, direct functional methods can be developed, an example of which is displayed in the Appendix J. This approach can be also exploited to show how to go ahead without fermions and so bypassing the potential problems related to breakdown of conformal anomaly since it allows a formulation where the ‘quarks’ are integrated out. Doing so apparently removes any conformal anomaly present in the boundary as we just discussed.

On the other hand, at some point we do expect a genuine breakdown of conformal invariance. This is so because of the conformal anomaly that the bosonic string theory must necessarily exhibit. In fact, this breakdown (tantamount to introducing a scale dependence) is quite welcome as we do not expect the simple string picture presented here to be even approximately true at very short distances (even within the large N limit). The naive bosonic string action used in the present paper does not prevent large Euclidean world sheets from crumpling³². It does not also describe correctly the high-temperature behavior of large N QCD³³.

According to by now common lore³⁴, the dependence on the conformal factor has to be understood in terms of the renormalization group flow. This dependence has not been worked out in detail, but it would be feasible to take it into account perturbatively in a way not too different from standard sigma model calculations in string theory by including dilaton, spin two, etc. backgrounds. The dilaton beta function precisely contains the $d - 26$ term characteristic of the anomaly. At some

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order in the α' and topological expansion there would appear a coupling between the dilaton and the chiral fields through the exchange of x string variables. It would be interesting how the hadronic degrees of freedom can modify the anomaly condition, if at all.

A QCD induced string action may also include nonlocal^{35,36,37} or, at least, higher-derivative³⁸ vertices breaking manifestly conformal symmetry which help to make the strings smooth and supply the correct high-temperature asymptotics. However, as we are concerned here with the low-energy string properties we do not expect that the strategy and technique to derive the chiral field action needs any significant changes to be adjusted to a modified QCD string action. The only changes may come out from the short distance behavior of the modified string propagator.

Finally we enumerate some of the simplifications and missing points of the approach undertaken in the present paper.

1) From the beginning we have restricted ourselves to the $SU(2)$ global flavor group. The reason is that only parity-even terms in the equations of motion can be revealed from the simple fermion Lagrangian (5). As it is supposed that all relevant meson degrees of freedom are reproduced by the hadronic string one cannot expect the appearance of multi-fermion interaction which may effectively arise only due to heavy-mass reduction of glueballs and hybrids suppressed in the large N limit. Then the only way to obtain the parity-odd Wess-Zumino-Witten terms is to supplement one-dimensional fermions with spinor degrees of freedom, i.e., for instance, to add reparameterization invariant vertex

$$\Delta L_f = \xi \bar{\psi} \gamma^\mu \dot{x}_\mu \psi, \quad (46)$$

with Dirac matrices γ^μ and new dimensional constant ξ . This extension will be investigated elsewhere.

2) One can easily add external electromagnetic fields and thereby calculate also the chiral constants L_9, L_{10} . However it is not yet clear how to introduce current quark masses consistently with the open string picture with ultra-relativistic quarks at their ends. Hence the other chiral constants need more efforts to be understood within the string approach if the boundary fermions were indeed associated with quarks.

3) Going back to the possibility of higher derivative³⁸ or nonlocal^{35,36,37} or five-dimensional^{39,34} action for the hadronic string, we find it interesting to explore the traces of deviations from the conformal string theory in low-energy chiral constants and phenomenology. Including the running of the $O(p^4)$ coefficients would however require the introduction of $1/N$ corrections. We do not know if the string picture can be consistently implemented when subleading corrections are taken into account.

4) As we have mentioned in this section —and emphasize here once more— the conformal anomaly can hopefully be taken into account consistently within the present approach, but we have not really attempted to do so in this present work. This is a major challenge that we leave for the future.

5) The value for the coefficients of $O(p^4)$ that we have found are, up to the usual factors of π , etc. rational numbers. In other words, they are not the first terms in the Taylor expansion of a Γ function or similar transcendental functions. What would be the appropriate amplitude replacing Veneziano formula then when the proper symmetries of the QCD vacuum are considered such as in the present case? We do not know.

All things considered, we believe that the objectives we set in the introduction have been achieved. We have obtained the leading effective action for low energy QCD from two very simple requirements: chiral invariance and conformal symmetry and two simple ideas: perturbing about the true vacuum of QCD and using the simplest possible effective world-sheet action. The approach seems to be oversimplified, but has proven to be considerably robust. All cross checks and physical requirements are met. The outcome is very sensible from a phenomenological point of view. Perhaps more importantly, several interesting avenues are open for future exploration and a fully consistent approach may be developed. We also hope that the considerations presented in this work may be of use to the ample string community since some of the techniques we have employed appear to be relevant in a wider context.

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Appendix A.

To construct the most general reparameterization- and chirally invariant action on the boundary of the string one can use the following set of operators bilinear in fermion variables and of the minimal dimension one

$$\begin{aligned} \bar{\psi}_L U \dot{\psi}_R, \quad \bar{\psi}_L U \psi_R, \quad \bar{\psi}_R U^+ \psi_L, \quad \bar{\psi}_R U^+ \dot{\psi}_L, \\ \bar{\psi}_L \dot{\psi}_L, \quad \bar{\psi}_R \dot{\psi}_R, \quad \bar{\psi}_L \dot{U} U^+ \psi_L, \quad \bar{\psi}_R U^+ \dot{U} \psi_R. \end{aligned} \quad (\text{A.1})$$

Other vertices like $\bar{\psi}_L \dot{U} \psi_R$ can be decomposed in a linear combination of basic vertices (A.1) after integration by parts in the action $S_b = \int d\tau L_f$. The multi-fermion local interaction is suppressed in the leading large N approximation as we assume that all meson degrees of freedom relevant in this limit are reproduced by the hadronic string. Then a multi-fermion interaction may effectively arise only due to heavy-mass reduction of glueballs, hybrids and multi-quark mesons suppressed in the large N limit.

As the case of CP symmetric action is of the most importance to provide the conformal symmetry we restrict ourselves with the analysis of this action only. Thus the general CP invariant hermitean Lagrangian takes the form:

$$\begin{aligned} L_f = b \bar{\psi}_L U \dot{\psi}_R + b^* \bar{\psi}_L U \psi_R + b^* \bar{\psi}_R U^+ \psi_L + b \bar{\psi}_R U^+ \dot{\psi}_L \\ + ic \left(\bar{\psi}_L \dot{\psi}_L + \bar{\psi}_R \dot{\psi}_R \right) + id \left(\bar{\psi}_L \dot{U} U^+ \psi_L - \bar{\psi}_R U^+ \dot{U} \psi_R \right), \end{aligned} \quad (\text{A.2})$$

where the constant b is complex whereas the constants c, d are real. As compared to the minimal Lagrangian (5) the general L_f contains three more real parameters, $\text{Im}(b), c, d$. Now let us show that by the local rotation of fermion variables preserving their chiral structure,

$$\begin{aligned} \psi_L = \alpha_1 \psi_L + \alpha_2 U \psi_R, \quad \bar{\psi}_L = \alpha_1^* \bar{\psi}_L + \alpha_2^* \bar{\psi}_R U^+, \\ \psi_R = \beta_1 \psi_R + \beta_2 U^+ \psi_L, \quad \bar{\psi}_R = \beta_1^* \bar{\psi}_R + \beta_2^* \bar{\psi}_L U, \end{aligned} \quad (\text{A.3})$$

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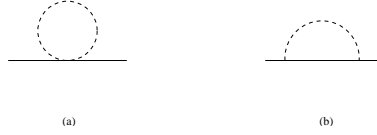


Fig. 1. One-loop diagrams for the propagator.

one can eliminate the redundant vertices and reduce the Lagrangian (5) to the minimal form. In order to prove it we transform the minimal Lagrangian (5) with the help of rotation (A.3). The initial set of constants is $b = (a + i)/2, c = d = 0$. The final set of vertices will be CP invariant if the following conditions are fulfilled:

$$\alpha_1 \beta_1^* = \alpha_1^* \beta_1, \quad (\text{A.4})$$

$$\alpha_2 \beta_2^* = \alpha_2^* \beta_2, \quad (\text{A.5})$$

$$\alpha_1 \beta_2^* = \alpha_2^* \beta_1. \quad (\text{A.6})$$

The first two constraints relate some phases,

$$\begin{aligned} \alpha_1 &= |\alpha_1| \exp(i\phi_1), & \beta_1 &= \pm |\beta_1| \exp(i\phi_1), \\ \alpha_2 &= |\alpha_2| \exp(i\phi_2), & \beta_2 &= \pm |\beta_2| \exp(i\phi_2), \end{aligned} \quad (\text{A.7})$$

whereas the third one eliminates one of the moduli of $|\alpha_j|$ and $|\beta_j|$. Three remaining moduli and the relative phase $\phi_1 - \phi_2$ turn out to be sufficient to fit three real constants $\text{Im}(b), c, d$,

$$\begin{aligned} |\alpha_1| |\beta_2| &= |\alpha_2| |\beta_1| = \frac{1}{2} \sqrt{c^2 + \frac{(2d+c)^2}{a^2}} \equiv \frac{1}{2} \zeta, \\ \cos(\phi_1 - \phi_2) &= \frac{c}{\zeta}, \\ \pm |\alpha_1| |\beta_1| \pm |\alpha_2| |\beta_2| &= \frac{1}{2} \zeta \left(\pm \frac{|\beta_1|}{|\beta_2|} \pm \frac{|\beta_2|}{|\beta_1|} \right) = \frac{1}{2} \text{Im}(b). \end{aligned} \quad (\text{A.8})$$

Evidently this system of equations has solutions for arbitrary $\text{Im}(b), c, d$ and therefore the minimal Lagrangian can be always obtained by an equivalence transformation (A.3) of fermion fields. At the quantum level this local transformation does not yield a nontrivial Jacobian when one applies the dimensional regularization to calculate it.

Appendix B.

In this and the following appendixes we present the results of our perturbative calculation. When necessary, both finite and divergent parts are given. Diagrams are labeled according to the figure number.

Diagram 1.a:

$$-\frac{1}{2} \theta(A - B) \Delta(0) U^{-1} \partial_\mu^2 U U^{-1}. \quad (\text{B.1})$$

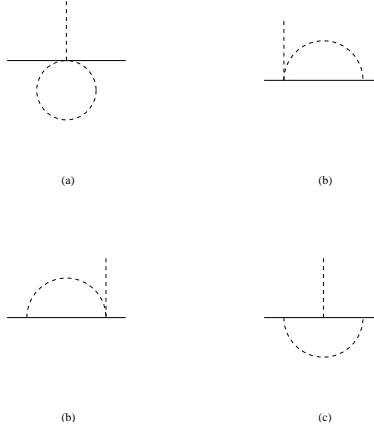
Fig. 2. One-loop diagrams for the vertex with one \tilde{x} -field.

Diagram 1.b:

$$\frac{1}{4}\theta(A-B) [(3+z^2)\Delta(0) + (1-z^2)\Delta(A,B)] U^{-1}\partial_\mu U U^{-1}\partial_\mu U U^{-1}. \quad (\text{B.2})$$

Appendix C.

In this appendix we show the result of the one-loop calculation of the vertex with two fermions and one boson line. The divergences that appear are not fully eliminated by a redefinition of U and additional counterterms with higher derivatives are called for.

Diagram 2.a:

$$-\frac{1}{4}\theta(A-B)\Delta(0)U^{-1}\partial_\mu(\partial^2 U)U^{-1} [\bar{x}_\mu(A) + \bar{x}_\mu(B) + z(\bar{x}_\mu(A) - \bar{x}_\mu(B))]. \quad (\text{C.1})$$

Diagram 2.b:

$$\begin{aligned} & \frac{1}{4}\theta(A-B)U^{-1} \{ [\Delta(0) (2(1+z)\bar{x}_\mu(A) + (1-z)^2\bar{x}_\mu(B)) \\ & + \Delta(A,B)(1-z^2)\bar{x}_\mu(B)] \partial_\nu U U^{-1}\partial_\mu U \\ & + [\Delta(0) ((1+z)^2\bar{x}_\mu(A) + 2(1-z)\bar{x}_\mu(B)) \\ & + \Delta(A,B)(1-z^2)\bar{x}_\mu(A)] \partial_\mu \partial_\nu U U^{-1}\partial_\nu U \} U^{-1}. \end{aligned} \quad (\text{C.2})$$

Diagram 2.c:

$$\begin{aligned} & -\frac{1}{8}\theta(A-B) \{ \Delta(0) [(1+z)^2(3-z)\bar{x}_\mu(A) + (1-z)^2(3+z)\bar{x}_\mu(B)] \\ & + \Delta(A,B)(1-z^2) [\bar{x}_\mu(A)(1+z) + \bar{x}_\mu(B)(1-z)] \\ & + (1-z^2) \int_B^A d\tau \dot{\bar{x}}_\mu(\tau) [\Delta(\tau, B)(1-z) - \Delta(A, \tau)(1+z)] \} \\ & \times U^{-1}\partial_\nu U U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}. \end{aligned} \quad (\text{C.3})$$

Appendix D.

This appendix has to do with the fermion determinant and the vanishing scale anomaly discussed in the main body of the paper.

There are two mutually conjugated operators in the bilinear form of the Lagrangian (5):

$$\begin{aligned}\mathcal{D} &= \frac{i}{2} \left[(1-z)U(\tau)i\partial_\tau + (1+z)i\partial_\tau \left(U(\tau) \right) \right], \\ \mathcal{D}^\dagger &= \frac{i}{2} \left[(1+z^*)U^+(\tau)i\partial_\tau + (1-z^*)i\partial_\tau \left(U^+(\tau) \right) \right].\end{aligned}\quad (\text{D.1})$$

Therefore the total fermion determinant (the result of integration over fermions) or fermion loop contribution can be represented by

$$Z_f = \|\mathcal{D}\mathcal{D}^\dagger\| = \left\| \left(i\partial_\tau - \frac{i}{2}(1-z)\dot{U}U^+ \right) \left(i\partial_\tau - \frac{i}{2}(1-z^*)\dot{U}U^+ \right) \right\|, \quad (\text{D.2})$$

where we have restricted ourselves with unitary fields U . Now one can factorize out the infinite constant for free operators and find the nontrivial part in terms of fermion propagators:

$$\begin{aligned}Z_f &= \|(i\partial_\tau)^2\| \exp \left\{ \text{Tr} \left(\log \left(1 - \frac{i}{2i\partial_\tau}(1-z)\dot{U}U^+ \right) \right. \right. \\ &\quad \left. \left. + \log \left(1 - \frac{i}{2i\partial_\tau}(1-z^*)\dot{U}U^+ \right) \right) \right\} \\ &= C \exp \left\{ -\theta(0) \frac{1}{2} \int_{-\infty}^{\infty} d\tau (1-z + 1-z^*) \text{tr}(\dot{U}U^+) \right\} \\ &= C \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \text{tr}(\dot{U}U^+) \right\}.\end{aligned}\quad (\text{D.3})$$

Herein the triangle property of the free fermion propagator has been exploited,

$$\langle \tau | \frac{i}{i\partial_\tau + i\epsilon} | \tau' \rangle = \theta(\tau - \tau'), \quad (\text{D.4})$$

which follows from the advanced Green function prescription by adding $+i\epsilon$. As a result, only the first order in the expansion of the logarithms in (D.3) survives when the functional trace operation is performed. The value $\theta(0) = 1/2$ and the CP invariant choice $z = -z^*$ are employed. More rigorously this result can be obtained by taking the finite proper-time interval with anti-periodic boundary conditions for fermion fields. Then the determinant will be given by the product of discrete eigenvalues of the operators (D.1). When proceeding to the infinite line limit and for the advanced prescription with $+i\epsilon$ one recovers exactly the functional presented in the last line of (D.3).

Evidently, for $SU(N)$ groups and for $U(1)$ groups with periodic boundary conditions the exponent in (D.3) vanishes and therefore the fermion loop contribution from the minimal Lagrangian is absent. It automatically eliminates the scale



Fig. 3. One-loop counterterms to the fermion propagator coming from the additional vertices.

anomaly $\sim \langle L_f \rangle_{vac}$ and thereby the conformal symmetry remains intact by vacuum polarization effects.

For the extended Lagrangian including the higher dimensional vertices (28) the derivation of the fermion determinant is similar. The corresponding differential operators look as follows:

$$\begin{aligned} \tilde{\mathcal{D}} &= \left(i\partial_\tau - \frac{i}{2}(1-z)\dot{U}U^+ + \frac{i}{8}(1-z^2) \left[(g_1 - zg_2)\partial_\nu \dot{U}U^+ \partial_\nu UU^+ (g_1 + zg_2) \right. \right. \\ &\quad \left. \left. - \partial_\nu UU^+ \partial_\nu \dot{U}U^+ + 2zg_3\partial_\nu UU^+ \dot{U}U^+ \partial_\nu UU^+ \right] \right) U, \\ \tilde{\mathcal{D}}^\dagger &= U^+ \left(i\partial_\tau - \frac{i}{2}(1-z^*)\dot{U}U^+ - \frac{i}{8}(1-(z^*)^2) \left[(g_1 - z^*g_2)U\partial_\nu U^+ U\partial_\nu \dot{U}^+ \right. \right. \\ &\quad \left. \left. - (g_1 + z^*g_2)U\partial_\nu \dot{U}^+ U\partial_\nu U^+ + 2z^*g_3U\partial_\nu U^+ U\dot{U}^+ U\partial_\nu U^+ \right] \right). \end{aligned} \quad (\text{D.5})$$

Again only the first order in the expansion of the logarithmic trace is not vanishing:

$$\begin{aligned} \log Z_f &= C' + \frac{1}{8} \int_{-\infty}^{\infty} d\tau \text{tr} \left((1-z^2) \left[zg_2\partial_\nu \dot{U}\partial_\nu U^+ + zg_3\partial_\nu UU^+ \dot{U}U^+ \partial_\nu UU^+ \right] \right. \\ &\quad \left. + (1-(z^*)^2) \left[-z^*g_2\partial_\nu \dot{U}\partial_\nu U^+ + z^*g_3\partial_\nu UU^+ \dot{U}U^+ \partial_\nu UU^+ \right] \right). \end{aligned} \quad (\text{D.6})$$

The vertices proportional g_1 do not appear after tracing. Let us take the CP symmetric constants $z = -z^*$. The vertices with the coupling constant g_3 happen to be proportional to $z + z^*$ and therefore vanish. As to the g_2 terms they form a total time derivative in the CP invariant case,

$$\log Z_f = C' + \frac{1}{8} \int_{-\infty}^{\infty} d\tau (1-z^2) z g_2 \partial_\tau \text{tr} (\partial_\nu U \partial_\nu U^+) = 0, \quad (\text{D.7})$$

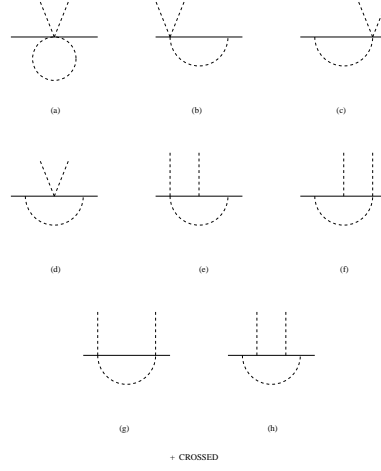
the latter taking place for periodic boundary conditions.

Thus, in spite of the fact that the higher dimensional vertices bring a scale dependence into the Lagrangian (the constants g_j are dimensional), they do not generate a scale anomaly due to fermion loops iff the CP symmetry is imposed on the Lagrangian.

Appendix E.

Here we list the contribution from the additional counterterms (26) to the fermion propagator.

Diagrams 3:

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 Fig. 4. One-loop diagrams for the vertex with two x -fields.

$$\frac{1}{4} (\Delta(0) - \Delta(A, B)) [(1+z)U^{-1}\partial_\mu U U^{-1}\phi_\mu U^{-1} - (1-z)U^{-1}\phi_\mu U^{-1}\partial_\mu U U^{-1}], \quad (\text{E.1})$$

where

$$\phi_\mu = \Delta(0) \frac{1-z^2}{2} \left(\frac{1-z}{2} \partial_\mu \partial_\nu U U^{-1} \partial_\nu U - \frac{1+z}{2} \partial_\nu U U^{-1} \partial_\mu \partial_\nu U + z \partial_\nu U U^{-1} \partial_\mu U U^{-1} \partial_\nu U \right). \quad (\text{E.2})$$

 When retaining only the terms $\sim \Delta^2(0)$ one reproduces exactly $2d_{II}$.

Appendix F.

Here we list the diagrams that are relevant to the calculation of the one-loop vertex involving two fermions and two boson lines.

Diagram 4.a:

$$-\frac{1}{8} \theta(A-B) \Delta(0) [\bar{x}_\mu(A) \bar{x}_\nu(A) (1+z) + \bar{x}_\mu(B) \bar{x}_\nu(B) (1-z)] U^{-1} \partial_\mu \partial_\nu \partial_\rho^2 U U^{-1}. \quad (\text{F.1})$$

Diagram 4.b:

Its divergent and finite parts are

$$\frac{1}{8} \theta(A-B) \left\{ \Delta(0) [\bar{x}_\mu(A) \bar{x}_\nu(A) 2(1+z) + \bar{x}_\mu(B) \bar{x}_\nu(B) (1-z)^2] + \Delta(A, B) \bar{x}_\mu(B) \bar{x}_\nu(B) (1-z^2) \right\} U^{-1} \partial_\rho U U^{-1} \partial_\mu \partial_\nu \partial_\rho U U^{-1}. \quad (\text{F.2})$$

Diagram 4.c:

Its divergent and finite parts are

$$\begin{aligned} & \frac{1}{8}\theta(A-B) \left\{ \Delta(0) [\bar{x}_\mu(A)\bar{x}_\nu(A)(1+z)^2 + \bar{x}_\mu(B)\bar{x}_\nu(B)2(1-z)] \right. \\ & \left. + \Delta(A,B)\bar{x}_\mu(A)\bar{x}_\nu(A)(1-z^2) \right\} U^{-1}\partial_\mu\partial_\nu\partial_\rho U U^{-1}\partial_\rho U U^{-1}. \end{aligned} \quad (\text{F.3})$$

Diagram 4.d:

Its divergent and finite parts are

$$\begin{aligned} & -\frac{1}{16}\theta(A-B) \left\{ \Delta(0) [\bar{x}_\mu(A)\bar{x}_\nu(A)(1+z)^2(3-z) + \bar{x}_\mu(B)\bar{x}_\nu(B)(1-z)^2(3+z)] \right. \\ & \left. + \Delta(A,B)(1-z^2) [\bar{x}_\mu(A)\bar{x}_\nu(A)(1+z) + \bar{x}_\mu(B)\bar{x}_\nu(B)(1-z)] \right. \\ & \left. + (1-z^2) \int_B^A d\tau (\bar{x}_\mu\dot{\bar{x}}_\nu(\tau) + \dot{\bar{x}}_\mu\bar{x}_\nu(\tau)) [\Delta(\tau,B)(1-z) - \Delta(A,\tau)(1+z)] \right\} \\ & \times U^{-1}\partial_\rho U U^{-1}\partial_\mu\partial_\nu U U^{-1}\partial_\rho U U^{-1}. \end{aligned} \quad (\text{F.4})$$

Diagram 4.e:

Its divergent and finite parts are

$$\begin{aligned} & -\frac{1}{8}\theta(A-B) \left\{ \Delta(0) [\bar{x}_\mu(A)\bar{x}_\nu(A)(1+z)^2(3-z) + \bar{x}_\mu(B)\bar{x}_\nu(B)(1-z)^2(3+z)] \right. \\ & \left. + (1-z)^2(1+z) \int_B^A d\tau \bar{x}_\mu(\tau)\dot{\bar{x}}_\nu(\tau) \right] \\ & \left. + \Delta(A,B)(1-z^2) [\bar{x}_\mu(A)\bar{x}_\nu(B)(1+z) + \bar{x}_\mu(B)\bar{x}_\nu(B)(1-z)] \right. \\ & \left. + (1-z^2) \int_B^A d\tau [\dot{\bar{x}}_\mu(\tau)\bar{x}_\nu(B)\Delta(\tau,B)(1-z) - \dot{\bar{x}}_\mu(\tau)\bar{x}_\nu(\tau)\Delta(A,\tau)(1+z)] \right\} \\ & \times U^{-1}\partial_\rho U U^{-1}\partial_\mu U U^{-1}\partial_\nu\partial_\rho U U^{-1}. \end{aligned} \quad (\text{F.5})$$

Diagram 4.f:

Its divergent and finite parts are

$$\begin{aligned} & -\frac{1}{8}\theta(A-B) \left\{ \Delta(0) [\bar{x}_\mu(A)\bar{x}_\nu(A)(1+z)^2(3-z) + \bar{x}_\mu(B)\bar{x}_\nu(B)(1-z)^2(3+z)] \right. \\ & \left. - (1-z)(1+z)^2 \int_B^A d\tau \dot{\bar{x}}_\mu(\tau)\bar{x}_\nu(\tau) \right] \\ & \left. + \Delta(A,B)(1-z^2) [\bar{x}_\mu(A)\bar{x}_\nu(A)(1+z) + \bar{x}_\mu(A)\bar{x}_\nu(B)(1-z)] \right. \\ & \left. + (1-z^2) \int_B^A d\tau [\bar{x}_\mu(\tau)\dot{\bar{x}}_\nu(\tau)\Delta(\tau,B)(1-z) - \bar{x}_\mu(A)\dot{\bar{x}}_\nu(\tau)\Delta(A,\tau)(1+z)] \right\} \\ & \times U^{-1}\partial_\rho\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U U^{-1}. \end{aligned} \quad (\text{F.6})$$

Diagram 4.g:

Its divergent and finite parts are

$$\frac{1}{8}\theta(A-B) \left\{ \Delta(0) [\bar{x}_\mu(A)\bar{x}_\nu(A)(1+z)(3+z)] + \bar{x}_\mu(B)\bar{x}_\nu(B)(1-z)(3-z) \right\}$$

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$$\begin{aligned}
& + (1 - z^2) \int_B^A d\tau (\bar{x}_\mu(\tau) \dot{x}_\nu(\tau) - \dot{x}_\mu(\tau) \bar{x}_\nu(\tau)) \Big] \\
& + 2\Delta(A, B)(1 - z^2) \bar{x}_\mu(A) \bar{x}_\nu(B) \Big\} \\
& \times U^{-1} \partial_\rho \partial_\mu U U^{-1} \partial_\rho \partial_\nu U U^{-1}. \tag{F.7}
\end{aligned}$$

Diagram 4.h:

Its divergent and finite parts are

$$\begin{aligned}
& \frac{1}{16} \theta(A - B) \left\{ \Delta(0) [\bar{x}_\mu(A) \bar{x}_\nu(A) 2(1 + z)^2 (3 - z) + \bar{x}_\mu(B) \bar{x}_\nu(B) 2(1 - z)^2 (3 + z)] \right. \\
& + \Delta(A, B)(1 - z^2) [\bar{x}_\mu(A) \bar{x}_\nu(A) 2(1 + z) + \bar{x}_\mu(B) \bar{x}_\nu(B)(1 - z)^2 \\
& \left. + \bar{x}_\mu(A) \bar{x}_\nu(B)(1 - z^2) - (1 - z^2) \int_B^A d\tau \dot{x}_\mu(\tau) \bar{x}_\nu(\tau) \right] \\
& + (1 - z^2) \int_B^A d\tau [\Delta(\tau, B) (\bar{x}_\mu(\tau) \dot{x}_\nu(\tau) 2(1 - z) \\
& + \dot{x}_\mu(\tau) \bar{x}_\nu(\tau)(1 - z^2) + \dot{x}_\mu(\tau) \bar{x}_\nu(B)(1 - z)^2) \\
& - \Delta(A, \tau) (\bar{x}_\mu(\tau) \dot{x}_\nu(\tau)(1 - z^2) + \dot{x}_\mu(\tau) \bar{x}_\nu(\tau) 2(1 + z) + \bar{x}_\mu(A) \dot{x}_\nu(\tau)(1 + z)^2) \\
& \left. - (1 - z^2)^2 \int_B^A d\tau_1 \int_B^{\tau_1} d\tau_2 \dot{x}_\mu(\tau_1) \dot{x}_\nu(\tau_2) \Delta(\tau_1, \tau_2) \right\} \\
& \times U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\rho U U^{-1}. \tag{F.8}
\end{aligned}$$

All divergences are removed by combining the renormalization of U and the additional counterterms determined from the one loop vertex with one external boson lines. No new counterterms are required.

Appendix G.

Two loop diagrams for the fermion propagator. Only the divergent parts are needed.

Diagram 5.a:

Its contribution is fully divergent,

$$-\frac{1}{8} \theta(A - B) \Delta^2(0) U^{-1} \partial_\mu^2 \partial_\rho^2 U U^{-1}. \tag{G.1}$$

Diagram 5.b:

Its divergent parts are

$$\frac{1}{8} \theta(A - B) \left\{ \Delta^2(0)(3 + z^2) + \Delta(A, B) \Delta(0)(1 - z^2) \right\} U^{-1} \partial_\mu^2 \partial_\rho U U^{-1} \partial_\rho U U^{-1}. \tag{G.2}$$

Diagram 5.c:

Its divergent parts are

$$\frac{1}{8} \theta(A - B) \left\{ \Delta^2(0)(3 + z^2) + \Delta(A, B) \Delta(0)(1 - z^2) \right\} U^{-1} \partial_\rho U U^{-1} \partial_\mu^2 \partial_\rho U U^{-1}. \tag{G.3}$$

Diagram 5.d:

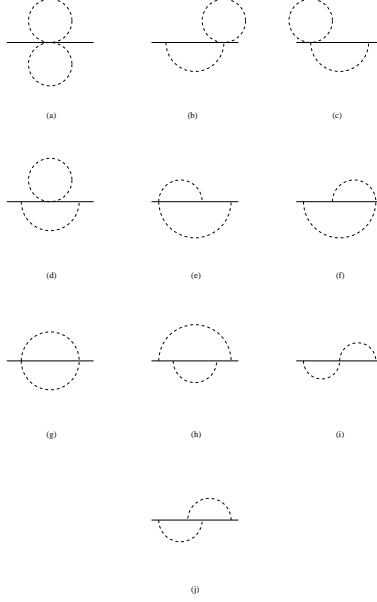


Fig. 5. Two-loop diagrams for the propagator.

Its divergent parts are

$$-\frac{1}{8}\theta(A-B)\{\Delta^2(0)(3+z^2)+\Delta(A,B)\Delta(0)(1-z^2)\} \\ \times U^{-1}\partial_\rho UU^{-1}\partial_\mu^2 UU^{-1}\partial_\rho UU^{-1}. \quad (\text{G.4})$$

Diagram 5.e:

Its divergent and finite parts are

$$-\frac{1}{16}\theta(A-B)\{\Delta^2(0)(11+z+5z^2-z^3)+\Delta(A,B)\Delta(0)2(1-z^2)(1-z) \\ +\Delta^2(A,B)(1-z^2)(3+z)\}U^{-1}\partial_\rho UU^{-1}\partial_\mu UU^{-1}\partial_\mu\partial_\rho UU^{-1}. \quad (\text{G.5})$$

Diagram 5.f:

Its divergent and finite parts are

$$-\frac{1}{16}\theta(A-B)\{\Delta^2(0)(11-z+5z^2+z^3)+\Delta(A,B)\Delta(0)2(1-z^2)(1+z) \\ +\Delta^2(A,B)(1-z^2)(3-z)\}U^{-1}\partial_\mu\partial_\rho UU^{-1}\partial_\mu UU^{-1}\partial_\rho UU^{-1}. \quad (\text{G.6})$$

Diagram 5.g:

Its divergent and finite parts are

$$\frac{1}{8}\theta(A-B)\{\Delta^2(0)(3+z^2)+\Delta^2(A,B)(1-z^2)\}U^{-1}\partial_\rho\partial_\mu UU^{-1}\partial_\rho\partial_\mu UU^{-1}. \quad (\text{G.7})$$

Diagram 5.h:

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Its divergent and finite parts are

$$\begin{aligned} & \frac{1}{32}\theta(A-B) \left\{ \Delta^2(0)(3+z^2)(7+z^2) + 2(1-z^2)(3+z^2)\Delta(A,B)\Delta(0) \right. \\ & + (1-z^2)(5-z^2)\Delta^2(A,B) \\ & \left. + 2(1-z^2)^2 \int_B^A d\tau_1 \int_B^{\tau_1} d\tau_2 \partial_{\tau_1} \Delta(\tau_1, \tau_2) \partial_{\tau_2} \Delta(\tau_1, \tau_2) \right\} \\ & \times U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\mu U U^{-1} \partial_\rho U U^{-1}. \end{aligned} \quad (\text{G.8})$$

Diagram 5.i

Its divergent and finite parts are

$$\begin{aligned} & -\frac{1}{2}\theta(A-B) \left[\Delta^2(0)(1+z^2) + \Delta(0)\Delta(A,B)(1-z^2) \right] \\ & \times U^{-1} \partial_\rho U U^{-1} \partial_\mu \partial_\rho U U^{-1} \partial_\rho U U^{-1}. \end{aligned} \quad (\text{G.9})$$

Diagram 5.j:

Its divergent and finite parts are

$$\begin{aligned} & \frac{1}{16}\theta(A-B) \left\{ \Delta^2(0)(1+z^2)(9-z^2) + 2(1-z^2)(3-z^2)\Delta(A,B)\Delta(0) \right. \\ & + (1-z^4)\Delta^2(A,B) - (1-z^2)^2 \int_B^A d\tau (\Delta(A, \tau) - \Delta(A, B)) \dot{\Delta}(\tau, B) \\ & \left. - (1-z^2)^2 \int_B^A d\tau_1 \int_B^{\tau_1} d\tau_2 \partial_{\tau_1} \Delta(\tau_1, \tau_2) \partial_{\tau_2} \Delta(\tau_1, \tau_2) \right\} \\ & \times U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1} \partial_\rho U U^{-1} \partial_\mu U U^{-1}. \end{aligned} \quad (\text{G.10})$$

The first integral is assembled into a non-singular expression. The second integral is prepared to have a weaker singularity than $\Delta^2(0)$ and it reveals a singularity of $\Delta(0)$ -type.

Appendix H.

Let us first understand the singularities in $J(A, B)$ with the help of Euclidean 2-dim scalar field propagator (15) in the cutoff regularization,

$$\Delta(\tau, \sigma) = -\alpha' \ln [(\tau^2 + \sigma^2 + R^2)\mu^2], \quad R \rightarrow 0. \quad (\text{H.1})$$

The regularization smears the log singularity, the normalization is taken to provide $-\partial^2 \Delta(\tau, \sigma) = 4\pi\alpha' \delta(\tau)\delta(\sigma)$ for $R \rightarrow 0$. Thus on the boundary,

$$\Delta(\tau, \sigma = 0) = -\alpha' \ln [(\tau^2 + R^2)\mu^2], \quad \partial_\sigma \Delta(\tau, \sigma = 0) = 0, \quad (\text{H.2})$$

the latter being in accordance with Neumann boundary conditions for open strings. Evidently, the relation between divergences in DR and the present regularization is:

$$\Delta(0) \simeq -\alpha' \ln [R^2 \mu^2] \leftrightarrow -\frac{2\alpha'}{\epsilon}. \quad (\text{H.3})$$

Of course this propagator is not necessarily exact in its finite part. But the divergences extracted with its help should be universal.

With this ansatz it is easy to find that

$$J(A - B) = \alpha'^2 \left\{ -\frac{\pi(A - B)}{R} - 2 \ln(R^2 \mu^2) \right\} + \text{regular terms.} \quad (\text{H.4})$$

The first divergence is power-like and we neglect it (it should not appear in DR calculation). The second term is logarithmically divergent $\simeq 2\alpha' \Delta(0)$ and we retain only this one.

Now let us perform the same calculation in the Dimensional Regularization. First we define the integral (39) in $2 + \epsilon$ dimensions. Evidently we can do it consistently for the second expression in (39). Namely, we replace $\tau \rightarrow |\vec{\tau}| \equiv t$ with $\vec{\tau}$ being a $1 + \epsilon$ dimensional vector and integrate over the sphere $|\vec{\tau}| \leq (A - B)$,

$$J_\epsilon(A - B) = -\frac{1}{2} \int_{|\vec{\tau}| \leq (A - B)} d^{1+\epsilon} \tau \mu^\epsilon (A - B - |\vec{\tau}|) \left[\dot{\Delta}(|\vec{\tau}|) \right]^2. \quad (\text{H.5})$$

Next we insert in (H.5) the derivative of the string propagator (15) in $2 + \epsilon$ dimensions,

$$\Delta'(t) = -\alpha' \epsilon \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{\mu\sqrt{\pi}}{\varphi}\right)^{-\epsilon} t^{-1-\epsilon}, \quad (\text{H.6})$$

which leads to

$$\begin{aligned} J_\epsilon(A - B) &= -\frac{1}{2} (\alpha')^2 \left(\epsilon \Gamma\left(\frac{\epsilon}{2}\right)\right)^2 \Omega_{1+\epsilon} \left(\frac{\mu\sqrt{\pi}}{\varphi}\right)^{-2\epsilon} \\ &\quad \times \int_0^{A-B} dt (t\mu)^\epsilon (A - B - t) t^{-2-\epsilon}, \end{aligned} \quad (\text{H.7})$$

where the angular volume,

$$\Omega_{1+\epsilon} = \frac{2\pi^{\frac{1+\epsilon}{2}}}{\Gamma\left(\frac{1+\epsilon}{2}\right)}. \quad (\text{H.8})$$

After integration one finds the following expression

$$\begin{aligned} J_\epsilon(A - B) &= -(\alpha' \varphi^{-\epsilon})^2 \left(\epsilon \Gamma\left(\frac{\epsilon}{2}\right)\right)^2 \frac{\sqrt{\pi}}{\Gamma\left(\frac{1+\epsilon}{2}\right)} \frac{((A - B)\mu\sqrt{\pi})^{-\epsilon}}{\epsilon(1 + \epsilon)} \\ &\stackrel{\epsilon \rightarrow 0}{=} -\frac{4(\alpha' \varphi^{-\epsilon})^2}{\epsilon} = 2\alpha' \varphi^{-\epsilon} \Delta(0). \end{aligned} \quad (\text{H.9})$$

Thus we have reproduced the same value of the constant c_V .

Appendix I.

Let us take the most general set of operators which can appear in the equations of motion (E.o.M.) with arbitrary constants. The equations of motion of dimension two (20) are assumed to hold and therefore we do not include any vertices containing

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the D’Alambertian ∂_μ^2 . We would like to find out a set of constants which supports the unitarity relation (21), $U\delta U^\dagger = -\delta U U^\dagger$. The results are presented in the Table 4.

Table 4. Comparison between the coefficients of the different chiral field structures in the equations of motion, their Hermitian conjugates and those ones derived from a local Lagrangian (42)

| CF structure | E.o.M. | (E.o.M.) [†] | χ -lagr. |
|------------------------------|--------|-------------------------|-------------------------|
| $\mu - -\rho - -\mu\rho$ | a_1 | $-a_2$ | $-2(2K_1 + K_2)$ |
| $\mu\rho - -\mu - -\rho$ | a_2 | $-a_1$ | $-2(2K_1 + K_2)$ |
| $\mu - -\mu\rho - -\rho$ | a_3 | $-a_3$ | $-4K_2$ |
| $\mu - -\rho - -\rho - -\mu$ | a_4 | $a_1 + a_2 + a_4$ | $2[(1 - z^2)K_1 + K_2]$ |
| $\mu - -\mu - -\rho - -\rho$ | a_5 | $a_3 + a_5$ | $-2z^2K_2$ |
| $\mu - -\rho - -\mu - -\rho$ | a_6 | $a_1 + a_2 + a_3 + a_6$ | $4[K_1 + K_2]$ |

One can see that the unitarity of $\delta U^{(4)}$ is provided for $z^2 = -1$ only if

$$\begin{aligned} a_1 = a_2 = -(4K_1 + 2K_2), \quad a_3 = -4K_2, \quad a_4 = -\frac{1}{2}(a_1 + a_2), \\ a_5 = -\frac{1}{2}a_3, \quad a_6 = -\frac{1}{2}(a_1 + a_2 + a_3). \end{aligned} \quad (\text{I.1})$$

Thus unitarity is achieved when the operators in the equation of motion are derived from the local Lagrangian (41) and *vice versa*.

Appendix J.

In this appendix we shall explore a somewhat different approach and see how the results are fully equivalent to those presented in the text. We shall integrate out the ‘quarks’ and derive an effective action in terms of external sources.

To derive the effective action let us supplement the Lagrangian (5) with the external sources for fermion fields

$$\tilde{L}_f = L_f + \bar{J}_L \psi_R + \bar{J}_R \psi_L + \bar{\psi}_L J_R + \bar{\psi}_R J_L. \quad (\text{J.1})$$

As this Lagrangian is quadratic in fields the effective action,

$$e^{iS_{eff}(x)} = \int d\bar{\psi} d\psi e^{i \int \tilde{L}_f dt} \quad (\text{J.2})$$

is supported by the solutions of classical equations,

$$\dot{\psi}_R + \frac{1}{2}(1+z)U^{-1}\dot{U}\psi_R = iU^{-1}J_R, \quad (\text{J.3})$$

$$\dot{\bar{\psi}}_R + \frac{1}{2}(1+z^*)\bar{\psi}_R\dot{U}^\dagger U^{\dagger-1} = -i\bar{J}_R U^{\dagger-1}, \quad (\text{J.4})$$

$$\dot{\psi}_L + \frac{1}{2}(1-z^*)U^{\dagger-1}\dot{U}^\dagger\psi_L = iU^{\dagger-1}J_L, \quad (\text{J.5})$$

$$\dot{\bar{\psi}}_L + \frac{1}{2}(1-z)\bar{\psi}_L\dot{U}U^{-1} = -i\bar{J}_L U^{-1}. \quad (\text{J.6})$$

The solutions read

$$\begin{aligned} \psi_R(\tau) = & \int_{-\infty}^{\tau} d\tau_1 \left[\mathbf{T} \exp \left(- \int_{\tau_1}^{\tau} d\tau_2 \frac{1}{2} (1+z) U^{-1}(\tau_2) \dot{U}(\tau_2) \right) \right] \\ & \times iU^{-1}(\tau_1) J_R(\tau_1), \end{aligned} \quad (\text{J.7})$$

$$\begin{aligned} \psi_L(\tau) = & \int_{-\infty}^{\tau} d\tau_1 \left[\mathbf{T} \exp \left(- \int_{\tau_1}^{\tau} d\tau_2 \frac{1}{2} (1-z^*) U^{\dagger-1}(\tau_2) \dot{U}^{\dagger}(\tau_2) \right) \right] \\ & \times iU^{\dagger-1}(\tau_1) J_L(\tau_1), \end{aligned} \quad (\text{J.8})$$

and their complex conjugated partners.

Therefore the effective action takes the simple form:

$$\begin{aligned} & \int_{-\infty}^{+\infty} d\tau \bar{J}_L(\tau) \int_{-\infty}^{\tau} d\tau_1 \left[\mathbf{T} \exp \left(- \int_{\tau_1}^{\tau} d\tau_2 \frac{1}{2} (1+z) U^{-1}(\tau_2) \dot{U}(\tau_2) \right) \right] \\ & \times iU^{-1}(\tau_1) J_R(\tau_1) \end{aligned} \quad (\text{J.9})$$

In turn the full fermion propagator takes the form

$$\begin{aligned} \langle \psi_R(\tau_1) \bar{\psi}_L(\tau_2) \rangle &= U^{-1}[x_{\mu}(\tau_1)] \mathbf{T} \exp \left[\frac{1}{2} (1-z) \int_{\tau_2}^{\tau_1} d\tau \dot{U} U^{-1}[x_{\mu}(\tau)] \right] \theta(\tau_1 - \tau_2), \\ &= \mathbf{T} \exp \left[-\frac{1}{2} (1+z) \int_{\tau_2}^{\tau_1} d\tau U^{-1} \dot{U}[x_{\mu}(\tau)] \right] \\ & \times U^{-1}[x_{\mu}(\tau_2)] \theta(\tau_1 - \tau_2). \end{aligned} \quad (\text{J.10})$$

We stress that two pieces with T-exponentials in the last equality are identical.

Let us expand the first expression for the propagator in (J.10)

$$\begin{aligned} \langle \psi_R(\tau_1) \bar{\psi}_L(\tau_2) \rangle &= \theta(\tau_1 - \tau_2) U^{-1}[x_{\mu}(\tau_1)] \left[1 + \frac{1}{2} (1-z) \int_{\tau_2}^{\tau_1} d\tau \dot{U} U^{-1}[x_{\mu}(\tau)] \right. \\ & \left. + \frac{1}{4} (1-z)^2 \int_{\tau_2}^{\tau_1} d\tau \int_{\tau_2}^{\tau} d\tau' \dot{U} U^{-1}[x_{\mu}(\tau)] \dot{U} U^{-1}[x_{\mu}(\tau')] + \dots \right], \end{aligned} \quad (\text{J.11})$$

The second order in the expansion is sufficient to analyze one-loop divergences of the propagator. In turn we can develop the perturbative expansion around a background, $x(\tau) = x_0 + \tilde{x}(\tau)$,

$$\begin{aligned} U^{-1}[x_{\mu}(\tau_1)] &= U^{-1}(x_0) \left\{ 1 - \tilde{x}_{\mu}(\tau_1) (\partial_{\mu} U) U^{-1}(x_0) \right. \\ & \left. + \frac{1}{2} \tilde{x}_{\mu}(\tau_1) \tilde{x}_{\nu}(\tau_1) [-(\partial_{\mu} \partial_{\nu} U) U^{-1}(x_0) + 2(\partial_{\mu} U) U^{-1}(\partial_{\nu} U) U^{-1}(x_0)] \right\}. \end{aligned} \quad (\text{J.12})$$

Evidently the term with two derivatives on U comes out only from this part of expansion. Further on one should also evaluate:

$$\begin{aligned} \dot{U} U^{-1}[x_{\mu}(\tau)] &= \dot{\tilde{x}}_{\mu} \partial_{\mu} U U^{-1}(x_0) + \dot{\tilde{x}}_{\mu} \tilde{x}_{\nu} [\partial_{\mu} \partial_{\nu} U U^{-1}(x_0) \\ & - \partial_{\mu} U U^{-1} \partial_{\nu} U U^{-1}(x_0)] + \dots \end{aligned} \quad (\text{J.13})$$

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Now we insert the above two expansions into Eq. (J.11) and retain only terms quadratic in \tilde{x}_μ ,

$$\begin{aligned}
& U^{-1}(x_0) \frac{1}{2} [-\partial_\mu \partial_\nu U U^{-1}(x_0) + 2\partial_\mu U U^{-1} \partial_\nu U U^{-1}(x_0)] \tilde{x}_\mu(\tau_1) \tilde{x}_\nu(\tau_1) \\
& - \frac{1}{2} (1-z) U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1}(x_0) \int_{\tau_2}^{\tau_1} d\tau \tilde{x}_\mu(\tau) \dot{\tilde{x}}_\nu(\tau) \\
& + \frac{1}{2} (1-z) U^{-1} [\partial_\mu \partial_\nu U U^{-1}(x_0) - \partial_\mu U U^{-1} \partial_\nu U U^{-1}(x_0)] \int_{\tau_2}^{\tau_1} d\tau \dot{\tilde{x}}_\mu(\tau) \tilde{x}_\nu(\tau) \\
& + \frac{1}{4} (1-z)^2 U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1}(x_0) \int_{\tau_2}^{\tau_1} d\tau \int_{\tau_2}^{\tau} d\tau' \dot{\tilde{x}}_\mu(\tau) \dot{\tilde{x}}_\nu(\tau') \quad (J.14)
\end{aligned}$$

After integration over τ, τ' and averaging in $\tilde{x}_\mu(\tau)$ with the help of formulas for the string propagator, having in mind that $\dot{\Delta}(0) = 0$ (i.e. the contribution of the third line is equal zero) one obtains the 1-loop part in the form written in the main text,

$$\begin{aligned}
& \frac{1}{2} \theta(\tau_1 - \tau_2) U^{-1} \left[\Delta(0) \left\{ -\partial_\mu^2 U \right. \right. \\
& \left. \left. + \frac{3+z^2}{2} \partial_\mu U U^{-1} \partial_\mu U \right\} + \frac{(1-z^2)}{2} \Delta(\tau_1 - \tau_2) \partial_\mu U U^{-1} \partial_\mu U \right] U^{-1}. \quad (J.15)
\end{aligned}$$